

ORLOV'S EQUIVALENCE AND TENSOR PRODUCTS: FROM SHEAVES TO MATRIX FACTORIZATIONS AND BACK

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ABSTRACT. A special case of a theorem due to Orlov states that for a hypersurface $X \subset \mathbb{P}^{n-1}$ of degree n given by the equation $W = 0$, there exists an equivalence between the bounded derived category $\mathbf{D}^b(\text{coh } X)$ of coherent sheaves on X and the homotopy category $\text{HMF}^{\text{gr}}(W)$ of graded matrix factorizations. We first give a description of this result, and present some methods for doing calculations with it. In the last two sections, we consider some possible product-like structures on these two triangulated categories, and provide possible directions for further inquiry as to what these product-like structures correspond to in either category, highlighting possible connections to tensor triangular geometry.

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INTRODUCTION

We let k be a field, and let B be a graded local k -algebra with $B_0 = k$. Fixing a non-zero homogeneous element $W \in B$, we can consider the quotient ring $A = B/(W)$; we restrict to the case when A has an isolated singularity at \mathfrak{m} .

A matrix factorization of W is defined to be a $\mathbb{Z}/2$ -graded complex of finite free B -modules such that $d^2 = \cdot W$. These form a $\mathbb{Z}/2$ -graded category $\mathrm{MF}^{\mathrm{gr}}(W)$, and the corresponding homotopy category forms the triangulated category $\mathrm{HMF}^{\mathrm{gr}}(W)$. This notion of a matrix factorization (in which the rings considered are not necessarily graded) first appeared in [Eis80, §5], in the context of studying the homological algebra of complete intersections. The study of matrix factorizations has since been studied in singularity theory, such as in [BGS87]. An overview of this work can be found in [Yos90]. It is surprising in recent years, however, that these objects have found relevance in physics as D -branes in Landau-Ginzburg models; their study is inspired by the Homological Mirror Symmetry Conjecture [Kon95].

The bounded derived category of coherent sheaves on $X = \mathrm{Proj}(A)$, denoted $\mathbf{D}^b(\mathrm{coh} X)$, on the other hand, has been studied more extensively because of its relevance to algebraic geometry, most notably in the context of Grothendieck duality [Nee10]. In [Orl09], expanding on work in [Orl04], Orlov proves the existence of the following commutative diagram of functors of triangulated categories, relating the two categories $\mathrm{HMF}^{\mathrm{gr}}(W)$ and $\mathbf{D}^b(\mathrm{coh} X)$:

$$(1) \quad \begin{array}{ccccc} & & \mathbf{D}^b(\mathrm{gr}\text{-}A) & & \mathrm{gr}\text{-}A \longleftarrow \mathrm{Cok} \text{---} \mathrm{MF}^{\mathrm{gr}}(W) \\ & \nearrow \mathbf{R}\Gamma_{\bullet} & \downarrow \pi^* & \searrow \gamma_i & \downarrow \delta_i \\ \mathbf{D}^b(\mathrm{coh} X) & \xrightarrow{\sim} & \mathbf{D}^b(\mathrm{qgr} A) & \xrightarrow{\sim} & \mathbf{D}_{\mathrm{Sg}}^{\mathrm{gr}}(A) \longleftarrow \mathrm{Cok}_i \text{---} \mathrm{HMF}^{\mathrm{gr}}(W) \\ & & & & \downarrow p \end{array}$$

where the bottom row are equivalences, and whose composition therefore gives the equivalence $\Psi_i: \mathbf{D}^b(\mathrm{coh} X) \xrightarrow{\sim} \mathrm{HMF}^{\mathrm{gr}}(W)$ between the bounded derived category of coherent sheaves to the homotopy category of matrix factorizations. Note the visual organization above is due to [Gal].

There are two main motivating questions for this paper:

- (1) Is there a way to calculate usual categorical invariants of $\mathbf{D}^b(\mathrm{coh} X)$ in terms of $\mathrm{HMF}^{\mathrm{gr}}(W)$?
- (2) What correspondence is there between usual functors on $\mathbf{D}^b(\mathrm{coh} X)$ with functors on $\mathrm{HMF}^{\mathrm{gr}}(W)$?

For our first question, even the Grothendieck group $K(\mathbf{D}^b(\mathrm{coh} X))$ is difficult to calculate. It is well-known that $K(\mathbf{D}^b(\mathrm{coh} X))$ is isomorphic to the Grothendieck group of the *abelian* category $\mathrm{coh} X$ [SGA5, VIII, §3], but this is still hard to calculate in general. For example, though, if X is a smooth elliptic curve, then $K(\mathbf{D}^b(\mathrm{coh} X)) \cong K(\mathrm{coh} X) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}$, where Pic denotes the Picard group of X [Har77, p. 149]. Another easy example is when X is affine, by Serre's equivalence $\mathrm{coh} X \cong A\text{-mod}$ [FAC, §49, Thm. 1], we have that $K(\mathbf{D}^b(\mathrm{coh} X)) \cong K(\mathrm{coh} X) \cong K(A\text{-mod})$, which is generated by the elements A/\mathfrak{p} for $\mathfrak{p} \in \mathrm{Spec}(A)$, and is equal to \mathbb{Z} in the case that A is a P.I.D. [AM69, p. 88]. It is hoped that eventually Orlov's equivalence (1) will provide a means to calculate Grothendieck groups via manipulating matrices in $\mathrm{HMF}^{\mathrm{gr}}(W)$, thereby reducing the question to a linear algebra question.

For our second question, [Gal] has provided a description of two functors in $\mathbf{D}^b(\mathrm{coh} X)$, namely, the dualizing functor \mathbb{D} and the Serre shift functor $- \otimes \mathcal{O}_X(\pm 1)$ in terms of matrix

factorizations. Note that the latter case corresponds to finding the action of the subgroup of the Picard group $\text{Pic}(X)$ generated by $\mathcal{O}_X(1)$, which at least in the case of a complete intersection of dimension ≥ 3 generates the whole Picard group [Har70, IV, Cor. 3.2]. Note that since in some cases like in those above, the Picard group gives substantial information about $K(\text{coh } X)$ (see [Man69, §10] for a precise description of how these two groups are related), we can hope that determining the action of $\mathcal{O}_X(1)$ would help us understand what $K(\text{coh } X) \cong K(\mathbf{D}^b(\text{coh } X))$ looks like.

Our present paper is an attempt to start understanding what kind of tensor product structure we can put on $\mathbf{D}^b(\text{coh } X)$ and $\text{HMF}^{\text{gr}}(W)$, and what they would correspond to under Orlov's equivalence Ψ_i , as an extension of the description of $-\otimes \mathcal{O}_X(1)$ in terms of matrix factorizations in [Gal].

1. TRIANGULATED CATEGORIES

We begin with some preliminaries about triangulated categories in general.

Definition 1.1. A *triangulated category* \mathcal{D} is an additive category with

- (a) an additive autoequivalence $[1]: \mathcal{D} \rightarrow \mathcal{D}$ called a *translation functor*, and
- (b) a class of *exact triangles* $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ which satisfy certain axioms (see [Ver96, II, Def. 1.1.1]).

We will only explicitly use the following axiom about exact triangles:

[T2]: A triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ of \mathcal{D} is exact if and only if the triangle $Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$ is exact.

The philosophy behind a triangulated category is that compared to an abelian category we have a different notion of an exact triple: instead of short exact sequences, we instead have exact triangles, with their own notion of an exact functor, namely, a functor $F: \mathcal{D} \rightarrow \mathcal{D}'$ that transforms exact triangles to exact triangles and commutes with $[1]$ [Ver96, II, Def. 1.1.3].

An easily accessible example of a triangulated category is the following:

Definition 1.2 ([Ver96, I, Def. 2.5.7]). Let \mathcal{A} be an abelian category. Then, the *homotopy category* $\mathbf{K}(\mathcal{A})$ is defined as the category of cochain complexes $\text{Kom}(\mathcal{A})$ with morphisms modulo chain homotopy. Similarly define the category $\mathbf{K}^b(\mathcal{A})$ obtained from the category of *bounded cochain complexes* $\text{Kom}^b(\mathcal{A})$.

The idea behind $\mathbf{K}(\mathcal{A})$ is that it takes chain complexes and makes them isomorphic if they are chain homotopy equivalent.

Theorem 1.3 ([Ver96, I, Prop. 1.3.2]). *The categories $\mathbf{K}(\mathcal{A}), \mathbf{K}^b(\mathcal{A})$ are triangulated.*

Despite the fact that $\mathbf{K}(\mathcal{A})$ is obtained from the *abelian* category $\text{Kom}(\mathcal{A})$, $\mathbf{K}(\mathcal{A})$ is not abelian. This motivates the need to define triangulated categories as something separate from abelian categories. To illustrate this, we provide the following example:

Example 1.4 ([HJ10, Ex. 2.6]). Consider the abelian category \mathbf{Ab} of abelian groups. Let $f: Y^\bullet \rightarrow Z^\bullet$ be the following morphism of complexes of abelian groups:

$$\begin{array}{cccccccc}
 \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \text{id} & & \downarrow & & \\
 \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots
 \end{array}$$

where the non-zero entries have degree $-1, 0$. In $\mathbf{Kom}(\mathbf{Ab})$ f is non-zero and has the zero complex as kernel. However, $f \simeq \text{id}$ with id as homotopy map, and so $f = 0$ in $\mathbf{K}(\mathbf{Ab})$.

We claim that f has no kernel (see [Wei94, Def. 1.2.1] for the universal property defining the kernel). Suppose f has a kernel, and so there exists a complex $X^\bullet := (\cdots \rightarrow X^{-1} \rightarrow X^0 \rightarrow X^1 \rightarrow \cdots)$ and a homomorphism $j = j^0: X^0 \rightarrow \mathbb{Z}$ ($j^i = 0$ for all $i \neq 0$ since Y^\bullet is concentrated at 0). Then, $\text{Im } j = k\mathbb{Z}$ for some $k \in \mathbb{Z}$ since it is a subgroup of \mathbb{Z} . Now consider $\ell: Y^\bullet \rightarrow X^\bullet$ given by multiplication by ℓ for any $\ell \in \mathbb{Z}$. $f \circ \ell = 0$ in $\mathbf{K}(\mathbf{Ab})$ since $f = 0$ in $\mathbf{K}(\mathbf{Ab})$. According to the universal property of the kernel, there must exist (unique) morphisms $u_\ell: \mathbb{Z} \rightarrow X^0$ such that $j \circ u_\ell \simeq \ell$. But since these are maps from Y^\bullet to X^\bullet and this complex is concentrated in degree 0, there are no non-zero homotopy maps, and so $j \circ u_\ell = \ell$ as group homomorphisms. But $\text{Im } j \circ u_\ell \subset \text{Im } j = k\mathbb{Z}$, so $j \circ u_\ell = \ell$ cannot hold for arbitrary $\ell \in \mathbb{Z}$, a contradiction.

Building on our construction for $\mathbf{K}(\mathcal{A})$, we can construct the category $\mathbf{D}(\mathcal{A})$ as follows. A subcategory \mathcal{C} of \mathcal{D} is *full* if it closed with respect to [1] and if \mathcal{C} contains any two objects of an exact triangle, the third is contained in \mathcal{C} as well. Now we recall the construction [Ver96, 2.1.7, 2.2.10] of the quotient category \mathcal{D}/\mathcal{C} of triangulated categories, where \mathcal{C} is a full subcategory of \mathcal{D} : two objects X, Y of \mathcal{D} are isomorphic if and only if they fit into a triangle $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ where Z is an object of \mathcal{C} . \mathcal{D}/\mathcal{C} is then a triangulated category by [Ver96, Thm. 2.2.6].

Definition 1.5 ([Ver96, III, 1.1.3]). Let $\mathbf{K}(\mathcal{A}), \mathbf{K}^b(\mathcal{A})$ as above. Then, define $\mathbf{Ac}(\mathcal{A}), \mathbf{Ac}^b(\mathcal{A})$ to be their respective full subcategories of acyclic complexes X^\bullet , i.e., complexes such that their cohomology groups $H^i(X^\bullet) = 0$ for all i .

Definition 1.6 ([Ver96, III, 1.1.3]). The *derived category* $\mathbf{D}(\mathcal{A})$ is defined as the quotient category $\mathbf{K}(\mathcal{A})/\mathbf{Ac}(\mathcal{A})$. Similarly define $\mathbf{D}^b(\mathcal{A})$ as the quotient $\mathbf{K}^b(\mathcal{A})/\mathbf{Ac}^b(\mathcal{A})$.

By definition and the discussion of quotient categories above, we have that

Theorem 1.7. *The categories $\mathbf{D}(\mathcal{A}), \mathbf{D}^b(\mathcal{A})$ are triangulated.*

Note that by quotienting out by $\mathbf{Ac}(\mathcal{A})$, what we are doing is taking all quasi-isomorphic cochain complexes in $\mathbf{D}(\mathcal{A})$ and making them isomorphic, in contrast with $\mathbf{K}(\mathcal{A})$, where only those quasi-isomorphisms that are also chain homotopy equivalences become isomorphisms.

2. DETAILED DESCRIPTION OF ORLOV'S EQUIVALENCE

We give a description of Orlov's equivalence when $B = k[[x_1, x_2, \dots, x_n]]$ with the standard grading, and fix $W \in B$ of degree n . Let A be the graded quotient ring $B/(W)$. We assume $\{W = 0\} \subset k^n$ has an isolated singularity at the origin. Note that we will try to introduce unfamiliar concepts on the way, in an attempt to make Orlov's work accessible to a wider audience.

2.1. Projective Schemes and Serre's Theorem. Consider the grading $A = \bigoplus_{d \geq 0} A_d$, and let A_+ be the ideal $\bigoplus_{d > 0} A_d$. We will follow the construction of the projective scheme $\text{Proj } A$ in [Har77, II, §2]. First, let

$$\text{Proj } A := \left\{ \underset{\text{set}}{\mathfrak{p}} \subseteq A \mid \begin{array}{l} \mathfrak{p} \text{ a prime, homogeneous ideal,} \\ \mathfrak{p} \not\subseteq A_+ \end{array} \right\}$$

Next, we define a topology on $\text{Proj } A$ with closed sets

$$V(\mathfrak{a}) := \{\mathfrak{p} \in \text{Proj } A \mid \mathfrak{p} \supset \mathfrak{a}\}$$

for each homogeneous ideal $\mathfrak{a} \subseteq A$. Finally, we define the sheaf of rings \mathcal{O} . Denote $A_{\mathfrak{p}} := T^{-1}A$, where T is the multiplicative system consisting of all homogeneous elements of A not in \mathfrak{p} . For any open subset $U \subseteq \text{Proj } A$, let $\mathcal{O}(U)$ be the set of functions $s: U \rightarrow \prod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$ which are locally fractions.

Definition 2.1. A *projective scheme* is the pair $(\text{Proj } A, \mathcal{O})$ for A a graded ring.

Note we usually denote a projective scheme just as $\text{Proj } A$.

There is a way to get sheaves on $\text{Proj } A$ from graded modules and vice versa (see [Har77, II, §5]). First, suppose M is a graded A -module. We construct the *sheaf associated with M* on $\text{Proj } A$, denoted by \widetilde{M} , as follows. For each $\mathfrak{p} \in \text{Proj } A$, let $M_{\mathfrak{p}}$ be the group of elements of degree zero in the localization $T^{-1}M$. For any open subset $U \subseteq \text{Proj } A$ let $\widetilde{M}(U)$ be the set of functions $s: U \rightarrow \prod_{\mathfrak{p} \in U} M_{\mathfrak{p}}$ that are locally fractions.

To go in the other direction, we need to use the sheaves associated to $A(n)$:

Definition 2.2. Let $X := \text{Proj } A$, A a graded ring. For any $n \in \mathbb{Z}$, we define $\mathcal{O}_X(n) := \widetilde{A(n)}$, where (n) denotes the grade shift $M(n)_i = M_{i+n}$. We call $\mathcal{O}_X(1)$ the *twisting sheaf* of Serre. For any sheaf of \mathcal{O}_X -modules \mathcal{F} , we define $\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

We collect a few facts about twisting sheaves here:

Proposition 2.3 ([Har77, II, Prop. 5.12]). *Let $X := \text{Proj } A$, A a graded ring. Assume A is generated by A_1 as an A_0 -algebra. Then, for any graded A -module M , $\widetilde{M}(n) \cong \widetilde{M(n)}$. In particular, $\mathcal{O}_X(m) \otimes \mathcal{O}_X(n) \cong \mathcal{O}_X(n+m)$.*

We can now define the *graded A -module associated to \mathcal{F}* when \mathcal{F} is a sheaf of \mathcal{O}_X -modules. As a group, let

$$\Gamma_{\bullet}(\mathcal{F}) := \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n)) \cong \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}(n)),$$

where $\Gamma(X, \mathcal{F})$ is the global sections functor, and we use the natural isomorphism $\Gamma(X, \mathcal{F}) \cong \text{Hom}(\mathcal{O}_X, \mathcal{F})$. We give it the structure of a graded S -module as follows: if $s \in S_d$, then s determines a global section $s \in \Gamma(X, \mathcal{O}_X(d))$, and so for any $t \in \Gamma(X, \mathcal{F}(n))$ we define the product $s \cdot t \in \Gamma(X, \mathcal{F}(n+d))$ by taking the tensor product $s \otimes t$ and using the natural map $\mathcal{F}(n) \otimes \mathcal{O}_X(d) \cong \mathcal{F}(n+d)$. Denote

$$\text{qgr } A := \text{gr-}A/\text{tors-}A$$

as the quotient category of $\text{gr-}A$ where objects are as in $\text{gr-}A$ and two objects M, N are isomorphic if and only if they fit into the exact sequence

$$0 \longrightarrow T \longrightarrow M \longrightarrow N \longrightarrow T' \longrightarrow 0$$

where both $T, T' \in \text{tors-}A$ (see [Pop73, §4.3]). Then, denote $\pi: \text{gr-}A \rightarrow \text{qgr } A$ as the natural projection functor. Recalling the definition of $\text{coh } X$ as the abelian category of coherent sheaves of finite type on X , we have the following theorem:

Theorem 2.4 ([FAC, §59, Prop. 7,8, §65, Prop. 6; EGAI, Prop. 3.3.5; AZ94, p. 229]). $\Gamma_{\bullet}: \text{coh } X \rightarrow \text{gr-}A$ is faithful, full, exact functor. Moreover, $\pi \circ \Gamma_{\bullet}$ is an equivalence, where $\pi: \text{gr-}A \rightarrow \text{qgr } A$ is the projection.

This equivalence can be visualized in the commutative diagram

$$(2) \quad \begin{array}{ccc} & & \text{gr-}A \\ & \nearrow \Gamma_{\bullet} & \downarrow \pi \\ \text{coh } X & \xrightarrow{\sim} & \text{qgr } A \end{array}$$

We would like to describe the right derived functor of Γ_{\bullet} .

$$\mathbf{R}\Gamma_{\bullet} : \mathbf{D}^b(\text{coh } X) \longrightarrow \mathbf{D}^b(\text{gr-}A).$$

where we review quickly the notion of a right derived functor (see [Har77, III, §1]). From [Har66, II, Thm. 7.18], we know that $\text{qcoh } X$ has enough injectives, i.e., every object in $\text{qcoh } X$ has an injective resolution, and so every quasi-coherent sheaf \mathcal{F} has an injective resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \dots$$

where \mathcal{I}^i are all injective objects in $\text{qcoh } X$, and where this sequence is exact. Note we use $\text{qcoh } X$ since $\text{coh } X$ does not have enough injectives. Since $\text{Hom}(\mathcal{O}_X, -)$ is left exact, we can apply it termwise to the complex $0 \rightarrow \mathcal{I}^{\bullet}$ to get the (not necessarily exact) complex of graded modules

$$\mathbf{R}\text{Hom}(\mathcal{O}_X, \mathcal{F}) := (0 \longrightarrow \text{Hom}(\mathcal{O}_X, \mathcal{I}^0) \longrightarrow \text{Hom}(\mathcal{O}_X, \mathcal{I}^1) \longrightarrow \dots)$$

in $\text{qgr } A$. We can therefore define the *right-derived functor* $\mathbf{R}\text{Hom}(\mathcal{O}_X, -)$ which gives the complex $\mathbf{R}\text{Hom}(\mathcal{O}_X, \mathcal{F})$ for a coherent sheaf \mathcal{F} . Likewise, we can define the right-derived functor $\mathbf{R}\Gamma_{\bullet}$ as the direct sum complex

$$(3) \quad \mathbf{R}\Gamma_{\bullet}(\mathcal{F}) := \bigoplus_{n \in \mathbb{Z}} \mathbf{R}\text{Hom}(\mathcal{O}_X, \mathcal{F}(n))$$

Note that this construction actually constructs a functor $\mathbf{D}^+(\text{qcoh } X) \rightarrow \mathbf{D}^+(\text{gr-}A)$. We therefore claim the following:

Theorem 2.5. $\mathbf{R}\Gamma_{\bullet}$ as defined in (3) defines a functor

$$\mathbf{R}\Gamma_{\bullet} : \mathbf{D}^b(\text{coh } X) \longrightarrow \mathbf{D}^b(\text{gr-}A),$$

such that we have the commutative diagram of functors

$$(4) \quad \begin{array}{ccc} & & \mathbf{D}^b(\text{gr-}A) \\ & \nearrow \mathbf{R}\Gamma_{\bullet} & \downarrow \pi^* \\ \mathbf{D}^b(\text{coh } X) & \xrightarrow{\sim} & \mathbf{D}^b(\text{qgr } A) \end{array}$$

where \sim denotes an equivalence and π^* is the functor induced by π on complexes.

Proof. First, we need to show $\mathbf{R}\Gamma_{\bullet}$ defines a functor $\mathbf{D}^b(\text{qcoh } X) \rightarrow \mathbf{D}^b(\text{gr-}A)$. By the vanishing theorem of Grothendieck [Har77, III, Thm. 2.7], the complex $\mathbf{R}\Gamma_{\bullet}(\mathcal{F})$ is bounded, and so (3) defines a functor $\mathbf{D}^b(\text{qcoh } X) \rightarrow \mathbf{D}^b(\text{gr-}A)$.

Next, we need to show (3) is a functor $\mathbf{D}^b(\text{coh } X) \rightarrow \mathbf{D}^b(\text{gr-}A)$. By [SGA6, II, Prop. 2.2.2], the inclusion $\mathbf{D}^b(\text{coh } X) \hookrightarrow \mathbf{D}^b(\text{qcoh } X)$ is full and faithful, and in fact realizes an equivalence $\mathbf{D}^b(\text{coh } X) \xrightarrow{\sim} \mathbf{D}^b(\text{qcoh } X)_{\text{coh}}$ where the latter is the full subcategory of all complexes with

coherent cohomologies. Thus, the definition (3) above actually gives a well-defined functor $\mathbf{D}^b(\text{coh } X) \rightarrow \mathbf{D}^b(\text{gr-}A)$.

Finally, by [Miy91, Thm. 3.2], we can “derive” the commutative diagram (2) to obtain the commutative diagram of derived categories (4). \square

2.2. Semiorthogonal Decompositions and Gorenstein Rings. We first define a new category $\mathbf{D}_{\text{Sg}}^{\text{gr}}(A)$, which we would like to show is equivalent to $\mathbf{D}^b(\text{qgr } A)$.

Let $\mathbf{D}^b(\text{grproj-}A)$ be the full subcategory of $\mathbf{D}^b(\text{gr-}A)$ consisting of objects isomorphic to bounded complexes of projectives. Recalling our construction of the quotient category in §1, we make the following definition:

Definition 2.6. The *triangulated category of singularities* is defined as

$$\mathbf{D}_{\text{Sg}}^{\text{gr}}(A) := \mathbf{D}^b(\text{gr-}A) / \mathbf{D}^b(\text{grproj-}A).$$

$\mathbf{D}_{\text{Sg}}^{\text{gr}}(A)$ is triangulated by construction as in §1.

We now give a short overview of the theory of semiorthogonal decompositions following [BK90, §1, §4; Or109, §1.1] used in proving the equivalence $\mathbf{D}^b(\text{qgr } A) \xrightarrow{\sim} \mathbf{D}_{\text{Sg}}^{\text{gr}}$.

Let \mathcal{D} be a triangulated category, and $\mathcal{N} \subset \mathcal{D}$ a full subcategory. The *right orthogonal* to \mathcal{N} is the full subcategory $\mathcal{N}^\perp \subset \mathcal{D}$ consisting of all objects M such that $\text{Hom}(N, M) = 0$ for all $N \in \mathcal{N}$. The *left orthogonal* ${}^\perp\mathcal{N}$ is defined similarly. Both $\mathcal{N}^\perp, {}^\perp\mathcal{N}$ are triangulated.

Definition 2.7. Let $\iota: \mathcal{N} \hookrightarrow \mathcal{D}$ be an embedding of a full triangulated subcategory \mathcal{N} in a triangulated category \mathcal{D} . \mathcal{N} is *right admissible* (resp. *left admissible*) if there is a right (resp. left) adjoint functor $q: \mathcal{D} \rightarrow \mathcal{N}$. \mathcal{N} is *admissible* if it is both right and left admissible.

We then have the following lemma relating the construction of the quotient category in §1 with orthogonal subcategories:

Lemma 2.8 ([BK90, Prop. 1.5, 1.6]). *Let \mathcal{N} be a right (resp. left) admissible subcategory of \mathcal{D} . Then, the quotient category \mathcal{N}^\perp (resp. ${}^\perp\mathcal{N}$) is equivalent to \mathcal{D}/\mathcal{N} . Conversely, if the quotient functor $q: \mathcal{D} \rightarrow \mathcal{D}/\mathcal{N}$ has a right (resp. left) adjoint, then \mathcal{D}/\mathcal{N} is equivalent to \mathcal{N}^\perp (resp. ${}^\perp\mathcal{N}$).*

Definition 2.9. A sequence of full triangulated subcategories $(\mathcal{N}_1, \dots, \mathcal{N}_n)$ in a triangulated category \mathcal{D} is called a *weak semiorthogonal decomposition* of \mathcal{D} if there is a sequence of left admissible subcategories $\mathcal{D}_1 = \mathcal{N}_1 \subset \mathcal{D}_2 \subset \dots \subset \mathcal{D}_n = \mathcal{D}$ such that \mathcal{N}_p is left orthogonal to \mathcal{D}_{p-1} in \mathcal{D}_p . If this is the case, we will write $\mathcal{D} = \langle \mathcal{N}_1, \dots, \mathcal{N}_n \rangle$.

By exhibiting a weak semiorthogonal decomposition for a triangulated category \mathcal{D} , what we are trying to do is decomposing \mathcal{D} into the simplest pieces \mathcal{N}_p possible.

We also have to review some facts about Gorenstein rings. Note that B is regular local by [AM69, p. 124], and so A is a complete intersection ring and thus Gorenstein by [Mat80, Thm. 21.3]. Moreover, A is a dualizing complex (see [Har66, V, §2]) over itself, as in the map $\mathbf{R}\text{Hom}(\mathbf{R}\text{Hom}(-, A), A)$ is an isomorphism, by [Har66, V, Ex. 2.2]. Then, we get two contravariant functors

$$\begin{aligned} \mathbb{D} &:= \mathbf{R}\text{Hom}_{\text{gr-}A}(-, A): \mathbf{D}^b(\text{gr-}A) \longrightarrow \mathbf{D}^b(\text{gr-}A^\circ), \\ \mathbb{D}^\circ &:= \mathbf{R}\text{Hom}_{\text{gr-}A^\circ}(-, A): \mathbf{D}^b(\text{gr-}A^\circ) \longrightarrow \mathbf{D}^b(\text{gr-}A), \end{aligned}$$

where $-\circ$ denotes the opposite category. Finally, we have that

Proposition 2.10 ([Orl09, Lem. 2.12, Prop. 2.16]). *The Gorenstein parameter of $A = B/(W)$, where $W \in B[[x_1, \dots, x_n]]$ has degree n is zero. Note here that the Gorenstein parameter is the integer a such that*

$$\mathbb{D}(k) = \mathbf{R} \operatorname{Hom}_{\operatorname{gr}\text{-}A}(k, A) \cong k(a)[-n]$$

Now we can return to our description of Orlov's equivalence. Denote by \mathcal{D}_i the subcategories of $\mathbf{D}^b(\operatorname{gr}\text{-}A)$ that are the images of the composition of functors

$$\mathbf{D}^b(\operatorname{coh} X) \xrightarrow{\mathbf{R}\Gamma_\bullet} \mathbf{D}^b(\operatorname{gr}\text{-}A) \xrightarrow{\operatorname{tr}_{\geq i}} \mathbf{D}^b(\operatorname{gr}\text{-}A_{\geq i}) \hookrightarrow \mathbf{D}^b(\operatorname{gr}\text{-}A)$$

where $\operatorname{tr}_{\geq i}$ is the exact functor defined as

$$\operatorname{tr}_{\geq i}(M^k)_j := \begin{cases} (M^k)_j & \text{if } j \geq i \\ 0 & \text{otherwise} \end{cases}$$

on each term M^k in a complex M^\bullet . By Theorem 2.4, all \mathcal{D}_i are equivalent to $\mathbf{D}^b(\operatorname{qgr} A)$.

Now for any i let $\mathcal{S}_{<i}(A)$ be the smallest full triangulated subcategory of $\mathbf{D}^b(\operatorname{gr}\text{-}A)$ containing the residue fields $k(j)$ for $j > -i$. Similarly define $\mathcal{S}_{\geq i}(A)$.

Furthermore, let $\mathcal{P}_{<i}(A)$ be the smallest full triangulated subcategory of $\mathbf{D}^b(\operatorname{gr}\text{-}A)$ containing the free modules $A(j)$ for $j > -i$. Similarly define $\mathcal{P}_{\geq i}(A)$.

Note we drop the dependence on A when it is clear which category these subcategories live in.

We then have the following semiorthogonal decompositions due to Orlov:

Lemma 2.11 ([Orl09, Lem. 2.3]). *The subcategories $\mathcal{S}_{<i}$ and $\mathcal{P}_{<i}$ are respectively left and right admissible in $\mathbf{D}^b(\operatorname{gr}\text{-}A)$ for all $i \in \mathbb{Z}$. Moreover, there are weak semiorthogonal decompositions*

$$\begin{aligned} \mathbf{D}^b(\operatorname{gr}\text{-}A) &= \langle \mathcal{S}_{<i}, \mathbf{D}^b(\operatorname{gr}\text{-}A_{\geq i}) \rangle & \mathbf{D}^b(\operatorname{tors}\text{-}A) &= \langle \mathcal{S}_{<i}, \mathcal{S}_{\geq i} \rangle \\ \mathbf{D}^b(\operatorname{gr}\text{-}A) &= \langle \mathbf{D}^b(\operatorname{gr}\text{-}A_{\geq i}), \mathcal{P}_{<i} \rangle & \mathbf{D}^b(\operatorname{grproj}\text{-}A) &= \langle \mathcal{P}_{\geq i}, \mathcal{P}_{<i} \rangle \end{aligned}$$

Lemma 2.12 ([Orl09, Lem. 2.4]). *The subcategories $\mathcal{S}_{\geq i}$ and $\mathcal{P}_{\geq i}$ are respectively right and left admissible in $\mathbf{D}^b(\operatorname{gr}\text{-}A)$ for all $i \in \mathbb{Z}$. Moreover, there are weak semiorthogonal decompositions*

$$\mathbf{D}^b(\operatorname{gr}\text{-}A_{\geq i}) = \langle \mathcal{D}_i, \mathcal{S}_{\geq i} \rangle \quad \mathbf{D}^b(\operatorname{gr}\text{-}A_{\geq i}) = \langle \mathcal{P}_{\geq i}, \mathcal{T}_i \rangle$$

where \mathcal{D}_i is equivalent to $\mathbf{D}^b(\operatorname{qgr} A)$ and \mathcal{T}_i is equivalent to $\mathbf{D}_{\operatorname{Sg}}^{\operatorname{gr}}(A)$.

With these two facts, we can prove the following theorem:

Theorem 2.13 ([Orl09, Thm. 2.5]). *There is an equivalence $\mathbf{D}^b(\operatorname{qgr} A) \xrightarrow{\sim} \mathbf{D}_{\operatorname{Sg}}^{\operatorname{gr}}(A)$ that fits into the commutative diagram of functors*

$$\begin{array}{ccc} \mathbf{D}^b(\operatorname{gr}\text{-}A) & & \\ \downarrow & \searrow^{\gamma_i} & \\ \mathbf{D}^b(\operatorname{qgr} A) & \xrightarrow{\sim} & \mathbf{D}_{\operatorname{Sg}}^{\operatorname{gr}}(A) \end{array}$$

where the functor γ_i is defined as the composition

$$\mathbf{D}^b(\operatorname{gr}\text{-}A) \xrightarrow{\operatorname{tr}_{\geq i}} \mathbf{D}^b(\operatorname{gr}\text{-}A) \xrightarrow{\mathbb{D}} \mathbf{D}^b(\operatorname{gr}\text{-}A^\circ) \xrightarrow{\operatorname{tr}_{\geq i-1}} \mathbf{D}^b(\operatorname{gr}\text{-}A) \xrightarrow{\mathbb{D}^\circ} \mathbf{D}^b(\operatorname{gr}\text{-}A).$$

Proof. Lemmas 2.11 and 2.12 give that \mathcal{T}_i is admissible in $\mathbf{D}^b(\text{gr-}A)$ and the right orthogonal \mathcal{T}_i^\perp has a weak semiorthogonal decomposition of the form

$$(5) \quad \mathcal{T}_i^\perp = \langle \mathcal{S}_{<i}, \mathcal{P}_{\geq i} \rangle.$$

We now describe the right orthogonal to \mathcal{D}_i . The dualizing functor \mathbb{D} takes the subcategory $\mathcal{S}_{\geq i}(A)$ to the subcategory $\mathcal{S}_{<-i+1}(A^\circ)$, and so sends the right orthogonal $\mathcal{S}_{\geq i}^\perp(A)$ to the left orthogonal ${}^\perp\mathcal{S}_{<-i+1}(A^\circ)$ which coincides with the right orthogonal $\mathcal{P}_{<-i+1}^\perp(A^\circ)$ by Lemma 2.11. Thus, the subcategory $\mathcal{S}_{\geq i}^\perp$ coincides with ${}^\perp\mathcal{P}_{<-i+1}(A^\circ)$. On the other hand, by Lemmas 2.11 and 2.12 we have

$${}^\perp\mathcal{P}_{\geq i}(A^\circ) = \mathcal{S}_{\geq i}^\perp \cong \langle \mathcal{S}_{<i}, \mathcal{D}_i \rangle,$$

which implies that the right orthogonal \mathcal{D}_i^\perp has the decomposition

$$\mathcal{D}_i^\perp = \langle \mathcal{P}_{\geq i}, \mathcal{S}_{<i} \rangle.$$

Now we have that the decomposition (5) is mutually orthogonal, for A is Gorenstein and $\mathbf{R}\text{Hom}_{\text{gr-}A}(k, A) = k[-n]$. Hence we can interchange $\mathcal{S}_{<i}, \mathcal{P}_{\geq i}$ to get

$$\mathcal{T}_i^\perp = \langle \mathcal{P}_{\geq i}, \mathcal{S}_{<i} \rangle.$$

Thus, $\mathcal{T}_i^\perp, \mathcal{D}_i^\perp$ are equal, and by Lemma 2.8 we have the equivalence $\mathbf{D}_{\text{Sg}}^{\text{gr}}(A) \cong \mathcal{T}_i \cong \mathcal{D}_i \cong \mathbf{D}^b(\text{qgr}A)$. The explicit description for γ_i follows from considering the steps above as functors, and in the proof of Lemma 2.12 in [Orl09]. \square

2.3. Graded Matrix Factorizations. We now finally construct the main triangulated category of interest, namely, that of graded matrix factorizations. Here we largely follow the expositions in [Tod, §2.2; Orl09, §3.1].

Definition 2.14. A *graded matrix factorization of W* is the data

$$(6) \quad P^\bullet := \left(P^0 \xrightarrow{p^0} P^1 \xrightarrow{p^1} P^0(d) \right)$$

where P^i are graded free B -modules of finite rank, and p^i are homomorphisms of graded B -modules satisfying

$$(7) \quad p^1 \circ p^0 = \cdot W, \quad p^0(d) \circ p^1 = \cdot W.$$

Note that (7) implies P^i have the same rank and p^i are square matrices.

We give a few elementary examples to illustrate the concept.

Example 2.15 ([Dyc11, Ex. 2.2]). Letting $B = \mathbb{C}[[x]]$ and $W = x^n$, we have the family of matrix factorizations parametrized by k :

$$B \begin{array}{c} \xrightarrow{x^k} \\ \xleftarrow{x^{n-k}} \end{array} B$$

Example 2.16 ([HW05, §4.1; Dyc11, Ex. 2.3]). Letting $B = \mathbb{C}[x, y, z]$, we would like to find the matrix factorization for $W = x^3 + y^3 + z^3 + \lambda xyz$ for $\lambda \in \mathbb{C}$. First consider the matrix

$$\varphi = \begin{pmatrix} ax & cy & bz \\ cz & bx & ay \\ by & az & cx \end{pmatrix}$$

φ then represents a linear map $B^3 \rightarrow B^3$. Since

$$\begin{aligned} \det \varphi &= abc(x^3 + y^3 + z^3) - (a^3 + b^3 + c^3)xyz = abcw \\ &\iff a^3 + b^3 + c^3 + \lambda abc = 0 \\ &\iff (a, b, c) \in \{W = 0\} \subset \mathbb{C}^3 \end{aligned}$$

if $(a, b, c) \in \{W = 0\} \subset (\mathbb{C}^*)^3$, then letting

$$\psi = \frac{1}{abc} \operatorname{adj}(\varphi) = \frac{1}{abc} \begin{pmatrix} bcx^2 - a^2yz & abz^2 - c^2xy & acy^2 - b^2xz \\ aby^2 - c^2xz & acx^2 - b^2yz & bcz^2 - a^2xy \\ acz^2 - b^2xy & bcy^2 - a^2xz & abx^2 - c^2yz \end{pmatrix}$$

where adj denotes the adjugate matrix, we see $\psi = W \cdot \varphi^{-1}$, and so $\varphi\psi = \psi\varphi = W \cdot I$.

Of course, we would like to find an easier method to calculate an arbitrary matrix factorization for a specific potential W . We therefore consider the category $\operatorname{HMF}^{\operatorname{gr}}(W)$ with objects as in (6):

Definition 2.17. The category of graded matrix factorizations of W , $\operatorname{MF}^{\operatorname{gr}}(W)$, is the category of objects (6), with morphisms given by the commutative diagram

$$\begin{array}{ccccc} P^0 & \xrightarrow{p^0} & P^1 & \xrightarrow{p^1} & P^0(d) \\ \downarrow f^0 & & \downarrow f^1 & & \downarrow f^0(d) \\ Q^0 & \xrightarrow{q^0} & Q^1 & \xrightarrow{q^1} & Q^0(d) \end{array}$$

Similarly, define the *homotopy category of graded matrix factorizations of W* , denoted as $\operatorname{HMF}^{\operatorname{gr}}(W)$, in the same way except with morphisms modulo null-homotopic morphisms. The above diagram is *null-homotopic* if there are homomorphisms of graded B -modules

$$h^0: P^0 \rightarrow Q^1(-d), \quad h^1: P^1 \rightarrow Q^0$$

satisfying

$$f^0 = q^1(-d) \circ h^0 + h^1 \circ p^0, \quad f^1 = q^0 \circ h^1 + h^0(d) \circ p^1.$$

We define a translation functor [1]:

$$P^\bullet[1] := \left(P^1 \xrightarrow{-p^1} P^0(d) \xrightarrow{-p^0(d)} P^1(d) \right)$$

Now for any morphism $f: P^\bullet \rightarrow Q^\bullet$, we can define the mapping cone $C^\bullet(f)$:

$$C^\bullet(f) := \left(P^0 \oplus Q^1 \xrightarrow{c^0} P^1 \oplus Q^0(d) \xrightarrow{c^1} P^0(d) \oplus Q^1(d) \right)$$

such that

$$c^0 = \begin{pmatrix} p^0 & f^1 \\ 0 & -q^1 \end{pmatrix}, \quad c^1 = \begin{pmatrix} p^1 & f^0 \\ 0 & -q^0 \end{pmatrix}$$

We define a standard triangle in $\operatorname{HMF}^{\operatorname{gr}}(W)$ as a triangle of the form

$$P^\bullet \xrightarrow{f} Q^\bullet \xrightarrow{(\operatorname{id}, 0)} C^\bullet(f) \xrightarrow{(0, -\operatorname{id})} K^\bullet[1]$$

for some $f: P^\bullet \rightarrow Q^\bullet$, and define a triangle in $\operatorname{HMF}^{\operatorname{gr}}(W)$ to be exact if it is isomorphic to a standard triangle. We then have the following theorem:

Theorem 2.18 ([Orl09, Prop. 3.4]). $\mathrm{HMF}^{\mathrm{gr}}(W)$ with the translation functor [1] and the class of exact triangles above forms a triangulated category.

The proof follows in the same way as for the usual homotopy category $K(\mathcal{A})$, proved in [GM03, Thm. IV.1.9].

There is then an equivalence of categories with $\mathbf{D}_{\mathrm{Sg}}^{\mathrm{gr}}(A)$ as defined before:

Theorem 2.19 ([Orl09, Prop. 3.5, Thm. 3.10]). *There exists a commutative diagram of functors*

$$\begin{array}{ccc} \mathrm{gr}\text{-}A & \xleftarrow{\mathrm{Cok}} & \mathrm{MF}^{\mathrm{gr}}(W) \\ \downarrow \delta_i & & \downarrow p \\ \mathbf{D}_{\mathrm{Sg}}^{\mathrm{gr}}(A) & \xleftarrow[\mathrm{Cok}_i]{\sim} & \mathrm{HMF}^{\mathrm{gr}}(W) \end{array}$$

such that Cok_i is an equivalence of categories and $p: \mathrm{MF}(W) \rightarrow \mathrm{HMF}(W)$ is the projection. Cok_i moreover sends a matrix factorization P^\bullet to the cokernel of p^0 , and in addition δ_i is defined as the composition

$$\mathrm{gr}\text{-}A \hookrightarrow \mathbf{D}^b(\mathrm{gr}\text{-}A) \xrightarrow{\mathrm{tr}_{\geq i}} \mathbf{D}^b(\mathrm{gr}\text{-}A) \xrightarrow{\mathbb{D}} \mathbf{D}^b(\mathrm{gr}\text{-}A^\circ) \xrightarrow{\mathrm{tr}_{\geq i-1}} \mathbf{D}^b(\mathrm{gr}\text{-}A) \xrightarrow{\mathbb{D}^\circ} \mathbf{D}^b(\mathrm{gr}\text{-}A)$$

where $\mathrm{gr}\text{-}A \hookrightarrow \mathbf{D}^b(\mathrm{gr}\text{-}A)$ puts the image of Cok in the zeroth degree.

We now describe the inverse Cok_i^{-1} of this functor following [Dyc11, §2.3] in the special case when our complex in $\mathbf{D}_{\mathrm{Sg}}^{\mathrm{gr}}(A)$ is zero everywhere except at index 0, where we have a graded module of the form B/I , I being generated by a regular sequence (f_1, \dots, f_m) , and $W \in I$. This gives the decomposition $W = \sum f_i w_i$ for $w_i \in B$.

Consider the Koszul complex associated to the sequence (f_1, \dots, f_m)

$$K = \left(\bigwedge^\bullet V, s_0 \right)$$

with V is the free module A^m with basis e_1, \dots, e_m , $\bigwedge^\bullet V$ denotes the exterior algebra over V , and s_0 denotes contraction with $(f_1, \dots, f_m) \in \mathrm{Hom}_A(V, B)$, i.e.,

$$s_0(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum_{k=1}^p (-1)^{k+1} f_{i_k} (e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_k} \wedge \cdots \wedge e_{i_p})$$

The complex K is then an B -free resolution of B/I by [Wei94, Cor. 4.5.5]. Then, multiplication by w on K is zero-homotopic, with the contracting homotopy s_1 given by exterior multiplication by $(w_1, \dots, w_m) \in V$, i.e.,

$$s_1(\omega) = \left(\sum_{i=1}^m w_i e_i \right) \wedge \omega$$

as shown in [BGS87, §2.2]. Since $s_0^2 = s_1^2 = 0$, the 2-periodic complex

$$\left(\bigoplus_{i=0}^m \bigwedge^\bullet V, s_0 + s_1 \right)$$

then defines a matrix factorization of w by [Dyc11, Cor. 2.7].

Example 2.20 ([Tod, §2.3]). Let $\mathfrak{m} = (x_1, \dots, x_n) \subset B$, and let $k(j) = (B/\mathfrak{m})(j)$. Then, the above construction gives us the matrix factorization

$$\begin{aligned} \bigoplus_{k \geq 0} \bigwedge^{2k+1} \mathfrak{m}/\mathfrak{m}^2 \otimes B(dk + j) &\xrightarrow{p^0} \bigoplus_{k \geq 0} \bigwedge^{2k} \mathfrak{m}/\mathfrak{m}^2 \otimes B(dk + j) \\ &\xrightarrow{p^1} \bigoplus_{k \geq 0} \bigwedge^{2k+1} \mathfrak{m}/\mathfrak{m}^2 \otimes B(d(k+1) + j) \end{aligned}$$

where p^0, p^1 are both defined by $s_0 + s_1$.

We note here for future reference that the Koszul complex above has another natural description in terms of the tensor product of matrix factorizations as will be defined later in §5.1, following [PV12, §2.2] and [Bec, §2.4]. Namely, denoting $\{a, b\}$ to be the matrix factorization of the potential $W = ab$ given as

$$B \xrightarrow{a} B \xrightarrow{b} B(d),$$

we see that the tensor product of these kind of matrix factorizations $\{f_i, w_i\}$ gives the Koszul complex above, since the Koszul complex itself can be described in terms of a tensor product of complexes of the form above [Wei94, §4.5]. Using this interpretation it is easier to see that different choices for w_i give isomorphic matrix factorizations [Bec, Rem. 2.4.4].

3. CALCULATIONS WITH ORLOV'S EQUIVALENCE

3.1. Calculations in General. Now that we know how each functor in the equivalence (1) works, we can start making computations with it. Note first, however, that at the current moment the equivalence depends on the choice of an integer $i \in \mathbb{Z}$. The following lemma makes this choice unimportant:

Lemma 3.1 ([Gal, Lem. 1.1]). $[1] \circ \Psi_i \circ (\mathcal{O}_X(-1) \otimes -) \cong \Psi_{i-1}$ for all $i \in \mathbb{Z}$

Proof. We first claim we have the following commutative diagram of functors:

$$\begin{array}{ccccccc} \mathbf{D}^b(\text{gr-}A) & \xrightarrow{\gamma_i} & \mathbf{D}_{\text{Sg}}^{\text{gr}}(A) & \xleftarrow{\delta_i} & \mathbf{D}^b(\text{gr-}A) & \xleftarrow{\text{Cok}} & \text{MF}^{\text{gr}}(W) \\ \downarrow [1] & & \downarrow [1] & & \downarrow [1] & & \downarrow [1] \\ & & & \swarrow \text{Cok}_i & & \swarrow p & \\ & & & \mathbf{HMF}^{\text{gr}}(W) & & & \\ & & & \downarrow [1] & & & \\ & & & \mathbf{HMF}^{\text{gr}}(W) & & & \\ & & & \swarrow \text{Cok}_{i-1} & & \swarrow p & \\ \mathbf{D}^b(\text{gr-}A) & \xrightarrow{\gamma_{i-1}} & \mathbf{D}_{\text{Sg}}^{\text{gr}}(A) & \xleftarrow{\delta_{i-1}} & \mathbf{D}^b(\text{gr-}A) & \xleftarrow{\text{Cok}} & \text{MF}^{\text{gr}}(W) \end{array}$$

$[1]: \mathbf{D}^b(\text{gr-}A) \rightarrow \mathbf{D}^b(\text{gr-}A)$ restricts to an equivalence $\mathbf{D}^b(\text{gr-}A_{\geq i}) \xrightarrow{\sim} \mathbf{D}^b(\text{gr-}A_{\geq i-1})$ sending $\mathcal{P}_{\geq i}$ to $\mathcal{P}_{\geq i-1}$ and $\mathcal{S}_{\geq i}$ to $\mathcal{S}_{\geq i-1}$, and $[1] \circ \text{tr}_{\geq i} \cong \text{tr}_{\geq i-1} \circ [1]$. Thus,

$$[1] \circ \gamma_i \cong \gamma_{i-1} \circ [1], \quad [1] \circ \delta_i \cong \delta_{i-1} \circ [1],$$

and so all of the squares commute, except possibly the trapezoid in the middle. But this commutes since p is a quotient functor, hence exact.

We now verify how $\mathbf{R}\Gamma_{\bullet}^{-1}$ and $[1]$ commute. Recall that $\mathbf{R}\Gamma_{\bullet}^{-1} = \widetilde{(-)}$ gives the associated sheaf for a graded module, and has the properties from Proposition 2.3. Thus,

$$\begin{aligned}
 [1] \circ \Psi_i \circ (\mathcal{O}_X(-1) \otimes -) \circ \widetilde{(-)} &\cong [1] \circ \Psi_i \circ \widetilde{(-)} \circ [-1] \\
 &\cong [1] \circ \text{Cok}_i^{-1} \circ \gamma_i \circ [-1] \\
 &\cong \text{Cok}_{i-1}^{-1} \circ [1] \circ \gamma_i \circ [-1] \\
 &\cong \text{Cok}_{i-1}^{-1} \circ \gamma_{i-1} \circ [1] \circ [-1] \\
 &\cong \text{Cok}_{i-1}^{-1} \circ \gamma_{i-1} \\
 &\cong \Psi_{i-1} \circ \widetilde{(-)}. \quad \square
 \end{aligned}$$

We therefore choose the equivalence

$$\Psi := \Psi_1 : \mathbf{D}^b(\text{coh } X) \rightarrow \text{HMF}^{\text{gr}}(W).$$

Following the lengthy description of Orlov's Equivalence in §2, we now have a step-by-step method to compute the matrix factorization for a given coherent sheaf as in [Gal, §1.2]:

- (1) Given a coherent sheaf \mathcal{F} , find $C \in \text{gr-}A$ such that $\widetilde{C} = \mathcal{F}$. We can do this by computing $\mathbf{R}\Gamma_{\bullet}(\mathcal{F})$ using sheaf cohomology as in [FAC], or just guessing.
- (2) Calculate $\gamma_1(C)$, which as we recall is given by

$$\mathbf{D}^b(\text{gr-}A) \xrightarrow{\text{tr}_{\geq i}} \mathbf{D}^b(\text{gr-}A) \xrightarrow{\mathbb{D}} \mathbf{D}^b(\text{gr-}A^\circ) \xrightarrow{\text{tr}_{\geq i-1}} \mathbf{D}^b(\text{gr-}A) \xrightarrow{\mathbb{D}^\circ} \mathbf{D}^b(\text{gr-}A).$$

which at worst requires finding two projective resolutions.

- (3) Use the process like in §2.3 to compute the inverse image through Cok_1 , and shift the resulting matrix factorization as needed.

Note that for this last step we have only developed a very special case; in general, however, it is not too difficult to calculate the inverse image using a computer program like Macaulay2 (see, for example, [Eis02, pp. 132–135]).

We now compute an example.

Example 3.2 ($\Psi(\mathcal{O}_X)$, following [Gal, Lem. 1.2]). We know that $\widetilde{A} = \mathcal{O}_X$, and so we first claim that $\gamma_1(A) \cong A_+$. Since $\text{tr}_{\geq 1}(A) = A_+$ by definition and since $\mathbb{D}^\circ \circ \mathbb{D} = \text{id}$, it suffices to show that $\mathbb{D}^\circ(A_+)$ is concentrated in degrees ≥ 0 . Using the long exact sequence for cohomology associated to the short exact sequence

$$0 \longrightarrow A_+ \longrightarrow A \longrightarrow k \longrightarrow 0$$

we see $\text{Ext}_{\text{gr}}^i(A_+, A) \cong \text{Ext}_{\text{gr}}^{i+1}(k, A)$ for all $i \geq 1$ and have an exact sequence

$$A \cong \text{Hom}_{\text{gr}}(A, A) \rightarrow \text{Hom}_{\text{gr}}(A_+, A) \rightarrow \text{Ext}_{\text{gr}}^1(k, A),$$

so the claim follows since A has Gorenstein parameter $a = 0$ by Proposition 2.10.

Now we have $\gamma_1(A) \cong A_+$ in $\mathbf{D}_{\text{Sg}}^{\text{gr}}$. Denoting A, A_+, k to also be the complexes in $\mathbf{D}_{\text{Sg}}^{\text{gr}}$ with A, A_+, k in degree 0, we have the exact triangle

$$A_+ \longrightarrow A \longrightarrow k \longrightarrow A_+[1].$$

By axiom [T2] for triangulated categories, this is exact if and only if

$$k[-1] \longrightarrow A_+ \longrightarrow A \longrightarrow k$$

is exact; however, since $A[1]$ is an object of $\text{grproj } A$, we see that $A_+ \cong k[-1]$ in $\mathbf{D}_{\text{Sg}}^{\text{gr}}$, and so it suffices to find $\text{Cok}_1^{-1}(k[-1]) \cong \text{Cok}_1^{-1}(k)[-1]$. But this we have done already in Example 2.20 for k , and so we simply shift everything by $[-1]$, i.e., we get the matrix factorization

$$\begin{aligned} \bigoplus_{k \geq 0} \bigwedge^{2k} \mathfrak{m}/\mathfrak{m}^2 \otimes B(d(k-1)) &\xrightarrow{-p^1(-d)} \bigoplus_{k \geq 0} \bigwedge^{2k+1} \mathfrak{m}/\mathfrak{m}^2 \otimes B(dk) \\ &\xrightarrow{-p^0} \bigoplus_{k \geq 0} \bigwedge^{2k} \mathfrak{m}/\mathfrak{m}^2 \otimes B(dk) \end{aligned}$$

3.2. Plane Elliptic Curves. Let $n = 3$, i.e., $B = \llbracket x_1, x_2, x_3 \rrbracket$, and let

$$W = x_1 \left(x_1^2 + \frac{\lambda}{3} x_2 x_3 \right) + x_2 \left(x_2^2 + \frac{\lambda}{3} x_1 x_3 \right) + x_3 \left(x_3^2 + \frac{\lambda}{3} x_1 x_2 \right).$$

Then, we have the matrix factorization corresponding to k

$$\begin{aligned} \langle e_1, e_2, e_3 \rangle \otimes B \oplus \langle e_1 \wedge e_2 \wedge e_3 \rangle \otimes B(3) \\ \xrightarrow{p^0} B \oplus \langle e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3 \rangle \otimes B(3) \\ \xrightarrow{p^1} \langle e_1, e_2, e_3 \rangle \otimes B(3) \oplus \langle e_1 \wedge e_2 \wedge e_3 \rangle \otimes B(6) \end{aligned}$$

where p^0, p^1 are given as above using the contraction map s_0 and external multiplication map s_1 . Denoting $x = x_1, y = x_2, z = x_3$, we then have the matrices

$$\begin{aligned} p^0 &= \begin{pmatrix} x & y & z & 0 \\ -y^2 - \frac{\lambda}{3}xz & x^2 + \frac{\lambda}{3}yz & 0 & z \\ -z^2 - \frac{\lambda}{3}xy & 0 & x^2 + \frac{\lambda}{3}yz & -y \\ 0 & -z^2 - \frac{\lambda}{3}xy & y^2 + \frac{\lambda}{3}xz & x \end{pmatrix}, \\ p^1 &= \begin{pmatrix} x^2 + \frac{\lambda}{3}yz & -y & -z & 0 \\ y^2 + \frac{\lambda}{3}xz & x & 0 & -z \\ z^2 + \frac{\lambda}{3}xy & 0 & x & y \\ 0 & z^2 + \frac{\lambda}{3}xy & -y^2 - \frac{\lambda}{3}xz & x^2 + \frac{\lambda}{3}yz \end{pmatrix} \end{aligned}$$

Performing the requisite $[-1]$ shift, we then have some degree shifts and have the matrices $q^0 := -p^1(-1)$ and $q^1 := -p^0$ as our matrix factorization corresponding to \mathcal{O}_X .

4. THE SEIDEL-THOMAS SHIFT FUNCTOR

In this section we recall the close relationship between the autoequivalence

$$[1]: \text{HMF}^{\text{gr}}(W) \rightarrow \text{HMF}^{\text{gr}}(W)$$

and the Seidel-Thomas twist (see [ST01]) on $\mathbf{D}^b(\text{coh } X)$, as an example of a situation in which the action of a functor on $\mathbf{D}^b(\text{coh } X)$ on $\text{HMF}^{\text{gr}}(W)$ through the equivalence

$$\Psi_i: \mathbf{D}^b(\text{coh } X) \xrightarrow{\sim} \text{HMF}^{\text{gr}}(W)$$

is well-known.

Recall that $\text{coh } X$ has a multiplicative structure given by the tensor product; in particular, we can construct the Picard group $\text{Pic}(X)$, which at least in the case of a projective scheme which is a complete intersection of dimension ≥ 3 is isomorphic to \mathbb{Z} with generator $\mathcal{O}_X(1)$ [Har70, IV, Cor. 3.2].

Recall that we have the following result as stated in [Tod, Prop. 3.2]:

Theorem 4.1 ([BFK12, Prop. 5.8]). *The following diagram commutes:*

$$\begin{array}{ccc} \mathbf{D}^b(\mathrm{coh} X) & \xrightarrow{\Psi_i} & \mathrm{HMF}^{\mathrm{gr}}(W) \\ \downarrow F_i & & \downarrow [1] \\ \mathbf{D}^b(\mathrm{coh} X) & \xrightarrow{\Psi_i} & \mathrm{HMF}^{\mathrm{gr}}(W) \end{array}$$

where $F_i := \mathrm{ST}_{\mathcal{O}_X(-i+1)} \circ (- \otimes \mathcal{O}_X(1))$ and ST is the Seidel-Thomas twist functor [ST01], defined as

$$\mathrm{ST}_{\mathcal{E}^\bullet}(*):= \mathrm{Cone}\left(\mathrm{hom}(\mathcal{E}^\bullet, *) \otimes \mathcal{E}^\bullet \xrightarrow{\mathrm{ev}} *\right)$$

where hom denotes the internal hom functor.

Following [Gal, p. 9], this gives the following isomorphism of functors:

$$(8) \quad \begin{aligned} \Psi_i \circ (- \otimes \mathcal{O}_X(1)) \circ \Psi_i^{-1} &\cong \mathrm{ST}_{\Psi_i(\mathcal{O}_X(-i+1))}^{-1} \circ [1] \\ \Psi_i \circ (- \otimes \mathcal{O}_X(-1)) \circ \Psi_i^{-1} &\cong [-1] \circ \mathrm{ST}_{\Psi_i(\mathcal{O}_X(-i+1))} \end{aligned}$$

We note that the Seidel-Thomas twist on the right side then has two arguments; we therefore define the following:

Definition 4.2. Define the bifunctor $[-, -]: \mathrm{HMF}^{\mathrm{gr}}(W) \times \mathrm{HMF}^{\mathrm{gr}}(W) \rightarrow \mathrm{HMF}^{\mathrm{gr}}(W)$ as

$$[Q^\bullet, P^\bullet] := \mathrm{ST}_{Q^\bullet}^{-1} P^\bullet [1].$$

Note this is an exact bifunctor by [ST01, Def. 2.5].

This gives rise to a couple of questions:

Question 4.3. What bifunctor does this induce on $\mathbf{D}^b(\mathrm{coh} X)$? It is clear that in the special case that $Q^\bullet = \Psi_i(\mathcal{O}_X(-i+1))$ for some i , we get that $[-, -]$ corresponds to $(- \otimes \mathcal{O}_X(1))$ on the $\mathbf{D}^b(\mathrm{coh} X)$ side by (8), but we do not know much otherwise. We speculate that it probably corresponds to something involving a tensor product.

Question 4.4. For what Q^\bullet does $[Q^\bullet, -]$ give an autoequivalence? This amounts to trying to figure out what it means for an object to be “spherical” in $\mathrm{HMF}^{\mathrm{gr}}(W)$ by [ST01, Prop. 2.10].

[Gal] gives a partial answer to the first question by describing the action of $- \otimes \mathcal{O}_X(\pm 1)$ in $\mathrm{HMF}^{\mathrm{gr}}(W)$; note that this would then fully describe the action of the Picard group in the case when X is a projective scheme which is a complete intersection of dimension ≥ 3 [Har70, IV, Cor. 3.2]. It is hoped that there is a concrete description of this action in terms of operations on matrices. We give an outline of his description below:

Recall (8), which we adapt to our specific choice of $i = 1$ in our equivalence $\Psi_i: \mathbf{D}^b(\mathrm{coh} X) \xrightarrow{\sim} \mathrm{HMF}^{\mathrm{gr}}(W)$:

$$\begin{aligned} \Psi \circ (- \otimes \mathcal{O}_X(1)) \circ \Psi^{-1} &\cong \mathrm{ST}_{\Psi(\mathcal{O}_X)}^{-1} \circ [1] \\ \Psi \circ (- \otimes \mathcal{O}_X(-1)) \circ \Psi^{-1} &\cong [-1] \circ \mathrm{ST}_{\Psi(\mathcal{O}_X)} \end{aligned}$$

Following [BFK12, Def. 2.28,2.29; Gal, p. 9; ST01, Def. 2.5,2.7], the Seidel-Thomas twist functor is given by calculating a k -basis

$$\{f_i\}_{i \in \mathbb{Z}} \subseteq \prod_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{HMF}^{\mathrm{gr}}}(\Psi(\mathcal{O}_X)[j], P^\bullet)$$

and then having

$$\mathrm{ST}_{\Psi(\mathcal{O}_X)}(P^\bullet) := \mathrm{Cone} \left(\bigoplus_{i \in \mathbb{Z}} \Psi(\mathcal{O}_X)[- \deg(f_i)] \longrightarrow P^\bullet \right)$$

where the map is equal to f_i on each summand. The inverse is described similarly by calculating a k -basis

$$\{g_i\}_{i \in \mathbb{Z}} \subseteq \prod_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{HMF}^{\mathrm{gr}}}(P^\bullet, \Psi(\mathcal{O}_X)[j])$$

and then having

$$\mathrm{ST}_{\Psi(\mathcal{O}_X)}(P^\bullet) := \mathrm{Cone} \left(P^\bullet \longrightarrow \bigoplus_{i \in \mathbb{Z}} \Psi(\mathcal{O}_X)[\deg(g_i)] \right)$$

where the map is equal to g_i on each summand.

By applying this process to find either ST or ST^{-1} , and applying the necessary shifts, in theory it would be possible to find the action the group generated by $\mathcal{O}_X(1)$ in $\mathrm{HMF}^{\mathrm{gr}}(W)$.

The original paper [BFK12] from which we derived the equivalences (8), however, does not give a hint as to what tensoring by an arbitrary sheaf on the $\mathbf{D}^b(\mathrm{coh} X)$ side will do on the matrix factorization side. Moreover, it seems as though their method does not generalize easily to an arbitrary tensor product in $\mathbf{D}^b(\mathrm{coh} X)$, and so a new method is needed.

5. TENSOR TRIANGULATED GEOMETRY AND $\mathrm{HMF}^{\mathrm{gr}}(W)$

We recall that there is a natural tensor product in $\mathrm{HMF}^{\mathrm{gr}}(W)$ inherited from its underlying structure as $\mathbb{Z}/2$ -graded complexes. This construction then has two variants listed below; again it would be nice to understand how these functors correspond on the coherent sheaf side through the equivalence Ψ_i .

Before we define these functors, we start with some preliminaries on *tensor triangulated geometry* (see [Bal10] for an overview).

Definition 5.1. A *tensor triangulated category* $(\mathcal{K}, \otimes, \mathbb{1})$ is a triangulated category \mathcal{K} with a monoidal structure (see [Mac98, VII])

$$- \otimes -: \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{K}$$

which unit object $\mathbb{1} \in \mathcal{K}$, such that for all $x \in \mathcal{K}$, the functors $x \otimes -$ and $- \otimes x$ are exact, and such that $a \otimes b \cong b \otimes a$ for all $a, b \in \mathcal{K}$.

Liu extends this definition to the graded case:

Definition 5.2 ([YL, §2.1]). An H -graded tensor triangulated category for H a commutative monoid is formed by a coproduct of tensor triangulated categories \mathcal{K}_α for $\alpha \in H$, i.e.,

$$\mathcal{K} := \prod_{\alpha \in H} \mathcal{K}_\alpha,$$

where objects are vectors $(x_\alpha)_{\alpha \in H}$ such that $x_\alpha = 0$ for almost all α , and

$$\mathrm{Hom}_{\mathcal{K}}((x_\alpha), (y_\alpha)) := \bigoplus_{\alpha \in H} \mathrm{Hom}_{\mathcal{K}_\alpha}(x_\alpha, y_\alpha).$$

The tensor product is defined such that for every pair $\alpha, \beta \in H$ we have the bifunctor

$$- \otimes_{\alpha\beta} -: \mathcal{K}_\alpha \times \mathcal{K}_\beta \rightarrow \mathcal{K}_{\alpha\beta}$$

that satisfies the usual properties for a graded product. Note that (x_α) is then the direct sum of all of its components, i.e., $(x_\alpha) = \bigoplus_\alpha x_\alpha$.

This category naturally has a tensor triangulated structure by defining [1] component-wise, and the tensor product between the component categories gives a tensor product on the whole category. Explicitly, we define $(x_\alpha) \otimes (y_\alpha) = (z_\alpha)$ such that

$$z_\alpha := \bigoplus_{\beta\gamma=\alpha} x_\beta \otimes y_\gamma.$$

The main application of these notions is due to Balmer:

Theorem 5.3 ([Bal10, Thm. 63, Rem. 64]). *Let X, Y be quasi-separated schemes. If their derived categories of quasi-coherent sheaves $\mathbf{D}(\mathrm{qcoh} X) \simeq \mathbf{D}(\mathrm{qcoh} Y)$ are equivalent as tensor triangulated categories then the schemes $X \cong Y$ are isomorphic.*

The main tool in [Bal05] is the definition of a *prime ideal* of a tensor triangulated category:

Definition 5.4 ([Bal05, Def. 1.2]). A *thick tensor-ideal* \mathcal{A} of \mathcal{K} is a full triangulated subcategory that is *thick*, i.e., for every object that decomposes into a direct sum, \mathcal{A} contains all the direct summands, and is a *tensor-ideal*, i.e., an ideal under \otimes in the commutative algebra sense.

Definition 5.5 ([Bal05, Def. 2.1]). A proper thick tensor-ideal \mathcal{P} is *prime* if $a \otimes b \in \mathcal{P} \implies a \in \mathcal{P}$ or $b \in \mathcal{P}$.

Using this definition, it is possible to define a *spectrum* $\mathrm{Spec}(\mathcal{K})$ on a triangulated category \mathcal{K} , quite analogously to the case with a prime ideal of a commutative ring.

In [YL], Liu applies this notion of a prime spectrum defined on a tensor triangulated category, to the graded case defined above. His main result is the following:

Theorem 5.6 ([YL, Thm. 3.3.3.2]). *If B is a Noetherian ring of finite Krull dimension then there is an isomorphism of locally ringed spaces*

$$\varphi: \mathrm{Spec}(B) \rightarrow \mathrm{Spec}(\mathrm{HMF}(0))$$

Note that in this case HMF denotes the homotopy category of ungraded matrix factorizations, in contrast to what we have been working with throughout.

5.1. The (Graded) Tensor Product in $\mathrm{HMF}^{\mathrm{gr}}(W)$. We would like to consider the possibility of a similar theorem to [YL, Thm. 3.3.3.2] for the graded triangulated category

$$\underline{\mathrm{HMF}}^{\mathrm{gr}}(W) := \coprod_{n \in \mathbb{N}} \mathrm{HMF}^{\mathrm{gr}}(nW)$$

where \mathbb{N} are the nonnegative integers. First, we must define what is meant by a tensor product in $\underline{\mathrm{HMF}}^{\mathrm{gr}}(W)$.

We follow [PV12, §2.2; YL, §3.1.1]. Let P^\bullet, Q^\bullet be in $\text{HMF}^{\text{gr}}(W), \text{HMF}^{\text{gr}}(W')$ respectively. Recall the notion of the *tensor product double complex* from [Wei94, §2.7], which is defined as the double complex

$$\begin{array}{ccccc}
P^0 \otimes Q^0(d) & \xrightarrow{p^0 \otimes \text{id}} & P^1 \otimes Q^0(d) & \xrightarrow{p^1 \otimes \text{id}} & P^0(d) \otimes Q^0(d) \\
\text{id} \otimes q^1 \uparrow & & -\text{id} \otimes q^1 \uparrow & & \text{id} \otimes q^1 \uparrow \\
P^0 \otimes Q^1 & \xrightarrow{p^0 \otimes \text{id}} & P^1 \otimes Q^1 & \xrightarrow{p^1 \otimes \text{id}} & P^0(d) \otimes Q^1 \\
\text{id} \otimes q^0 \uparrow & & -\text{id} \otimes q^0 \uparrow & & \text{id} \otimes q^0 \uparrow \\
P^0 \otimes Q^0 & \xrightarrow{p^0 \otimes \text{id}} & P^1 \otimes Q^0 & \xrightarrow{p^1 \otimes \text{id}} & P^0(d) \otimes Q^0
\end{array}$$

We can then define the *tensor product chain complex* as the total chain complex Tot^\oplus obtained from this double complex (see [Wei94, §1.2]).

Definition 5.7 (cf. [GK, 1.6.8]). Given two matrix factorizations P^\bullet, Q^\bullet in the triangulated categories $\text{HMF}^{\text{gr}}(W)$ and $\text{HMF}^{\text{gr}}(W')$ respectively, the *tensor product* $P^\bullet \otimes Q^\bullet$ is defined as the matrix factorization in $\text{HMF}^{\text{gr}}(W + W')$ defined by the tensor product chain complex above, explicitly given as

$$P^\bullet \otimes Q^\bullet := \left(\begin{array}{ccc}
P^0 \otimes Q^0 & \begin{pmatrix} p^0 \otimes \text{id}(d) & -\text{id} \otimes q^1 \\ \text{id}(d) \otimes q^0 & p^1 \otimes \text{id} \end{pmatrix} & P^1 \otimes Q^0(d) \\
\oplus & \xrightarrow{\quad} & \oplus \\
P^1 \otimes Q^1 & & P^0(d) \otimes Q^1 \\
& & \begin{pmatrix} p^1 \otimes \text{id} & \text{id} \otimes q^1 \\ -\text{id}(d) \otimes q^0 & p^0 \otimes \text{id}(d) \end{pmatrix} & P^0(d) \otimes Q^0(d) \\
& & \oplus & \\
& & P^1(d) \otimes Q^1(d) &
\end{array} \right)$$

where we will sometimes drop the degree shifts (d) for clarity.

It is clear that this tensor product \otimes defines a tensor structure on $\underline{\text{HMF}}^{\text{gr}}(W)$ with $H = \mathbb{N}$ as our commutative monoid. This gives us our first question:

Question 5.8. What bifunctor does

$$- \otimes -: \text{HMF}^{\text{gr}}(mW) \times \text{HMF}^{\text{gr}}(nW) \rightarrow \text{HMF}^{\text{gr}}((m+n)W)$$

induce on $\coprod_{n \in \mathbb{Z}} \mathbf{D}^b(\text{coh } X_n)$, where the $X_n := \{nW = 0\}$?

Note that the action of the tensor product is qualitatively different in zero and prime characteristic, for the tensor product provides a multiplicative structure on

$$\underline{\text{HMF}}^{\text{gr}}(W) := \coprod_{n \in \mathbb{Z}/(p)} \text{HMF}^{\text{gr}}(nW)$$

for prime characteristic p .

First we show a couple of facts concerning properties of $\underline{\text{HMF}}^{\text{gr}}(W)$ intrinsically, so that instead of using the abstract construction of a prime spectra in [Bal05], we can instead use the familiar construction from commutative algebra.

Lemma 5.9. $- \otimes -$ is commutative, i.e., $P^\bullet \otimes Q^\bullet = Q^\bullet \otimes P^\bullet$ (up to isomorphism).

Proof. This follows by considering the canonical isomorphisms $P^i \otimes Q^j \cong Q^j \otimes P^i$ for graded modules and taking direct sums to get an isomorphism $P^\bullet \otimes Q^\bullet \cong Q^\bullet \otimes P^\bullet$. \square

Proposition 5.10 ([YL, §3.1.2]). *The identity element in $\underline{\text{HMF}}^{\text{gr}}(W)$ with respect to \otimes is given by the matrix factorization*

$$U_1^\bullet := \left(B \xrightarrow{0} 0 \xrightarrow{0} B \right)$$

Proof. Let P^\bullet be another matrix factorization in $\text{HMF}^{\text{gr}}(nW)$ for some $n \in \mathbb{Z}$. Then, the tensor product $P^\bullet \otimes U^\bullet$ is given by

$$\begin{array}{ccc} P^0 \otimes B & \begin{pmatrix} p^0 \otimes \text{id}(d) & -\text{id} \otimes 0 \\ \text{id}(d) \otimes 0 & p^1 \otimes \text{id} \end{pmatrix} & P^1 \otimes B & \begin{pmatrix} p^1 \otimes \text{id} & \text{id} \otimes 0 \\ -\text{id}(d) \otimes 0 & p^0 \otimes \text{id}(d) \end{pmatrix} & P^0(d) \otimes B \\ \oplus & \xrightarrow{\quad} & \oplus & \xrightarrow{\quad} & \oplus \\ P^1 \otimes 0 & & P^0(d) \otimes 0 & & P^1(d) \otimes 0 \end{array}$$

and using the canonical isomorphisms $M \otimes B \cong P^0$, $N \otimes 0 \cong 0$ this is isomorphic to P^\bullet . \square

Recalling the notion of the *direct sum* of matrix factorizations

$$P^\bullet \oplus Q^\bullet := \left(P^0 \oplus Q^0 \xrightarrow{\begin{pmatrix} p^0 & \\ & q^0 \end{pmatrix}} P^1 \oplus Q^1 \xrightarrow{\begin{pmatrix} p^1 & \\ & q^1 \end{pmatrix}} P^0(d) \oplus Q^0(d) \right)$$

and defining $nP^\bullet := \bigoplus_n P^\bullet$, we have the following corollary:

Corollary 5.11. *The isomorphism classes of objects in $\underline{\text{HMF}}^{\text{gr}}(W)$ form a \mathbb{Z} -module with \mathbb{Z} -action defined by $- \cdot i := - \otimes U_i^\bullet$, where U_i^\bullet is defined by*

$$U_i^\bullet := \left(B^i \xrightarrow{0} 0 \xrightarrow{0} B^i \right)$$

Proof. The argument is as in Proposition 5.10, where we use the canonical isomorphism $M \otimes B^i \cong \oplus_i M \otimes B \cong \oplus_i M$ [Mat80, p. 327]. \square

This, however, is a *very* special case of the following:

Theorem 5.12. *The isomorphism classes of objects in $\underline{\text{HMF}}^{\text{gr}}(W)$ with multiplication defined by \otimes and addition defined by \oplus form a graded commutative ring with identity.*

Proof. Proposition 5.10 shows there is an identity, Corollary 5.11 shows \oplus gives an abelian group structure, and \otimes gives a monoidal structure by the fact that it gives a monoidal structure on $B\text{-mod}$. Distributivity follows similarly by the distributivity of \oplus, \otimes on the level of B -modules [Mat80, p. 327]. \square

What is notable about this theorem, then, is that there is now a distinct difference between our approach and that of [Bal05]: since $\underline{\text{HMF}}^{\text{gr}}(W)$ forms a graded ring, then we think that instead of the *affine* spectrum for triangulated categories defined in [Bal05], it would be more natural for us to use the *projective* version $\text{Proj}(\underline{\text{HMF}}^{\text{gr}}(W))$.

5.2. The ‘‘Internal’’ Tensor Product in $\text{HMF}^{\text{gr}}(W)$. In contrast to the graded case above, it is also possible to define a tensor product $- \otimes -: \text{HMF}^{\text{gr}}(W) \otimes \text{HMF}^{\text{gr}}(W) \rightarrow \text{HMF}^{\text{gr}}(W)$. This is in order to prove something like the following conjecture:

Conjecture 5.13. The image of the Picard group through Ψ forms a group in HMF^{gr} under the group operation \otimes defined above.

and to find the answer to the following question:

Question 5.14. What are the “ \otimes -invertible objects” in $\mathrm{HMF}^{\mathrm{gr}}(W)$? Namely, what are the objects $P^\bullet \in \mathrm{HMF}^{\mathrm{gr}}(W)$ for which there exists $Q^\bullet \in \mathrm{HMF}^{\mathrm{gr}}(W)$ with $P^\bullet \otimes Q^\bullet \cong \mathbb{1}$ (cf. [Bal10, Def. 35])? Note that this is equivalent to asking for which objects P^\bullet do we have an autoequivalence $P^\bullet \otimes - : \mathrm{HMF}^{\mathrm{gr}}(W) \rightarrow \mathrm{HMF}^{\mathrm{gr}}(W)$.

Conjecture 5.13 as-is, however, makes no sense because

$$- \otimes - : \mathrm{HMF}^{\mathrm{gr}}(W) \otimes \mathrm{HMF}^{\mathrm{gr}}(W) \rightarrow \mathrm{HMF}^{\mathrm{gr}}(2W)$$

We intend to pursue Conjecture 5.13 in the following, by redefining the tensor product so it lands in $\mathrm{HMF}^{\mathrm{gr}}(W)$. Because $2W = 0$ if $\mathrm{char} k = 2$, we add the assumption that $\mathrm{char} k \neq 2$ to make the tensor product well-defined in $\mathrm{HMF}^{\mathrm{gr}}(W)$.

Recalling that $B = k[[x_1, x_2, \dots, x_n]]$, we can redefine the tensor product as the bifunctor $- \otimes - : \mathrm{HMF}^{\mathrm{gr}}(W) \otimes \mathrm{HMF}^{\mathrm{gr}}(W) \rightarrow \mathrm{HMF}^{\mathrm{gr}}(W)$ where

$$P^\bullet \otimes Q^\bullet := \left(\begin{array}{c} P^0 \otimes Q^0 \\ \oplus \\ P^1 \otimes Q^1 \end{array} \xrightarrow{\frac{1}{2} \begin{pmatrix} p^0 \otimes \mathrm{id} & -\mathrm{id} \otimes q^1 \\ \mathrm{id} \otimes q^0 & p^1 \otimes \mathrm{id} \end{pmatrix}} \begin{array}{c} P^1 \otimes Q^0 \\ \oplus \\ P^0 \otimes Q^1 \end{array} \xrightarrow{\begin{pmatrix} p^1 \otimes \mathrm{id} & \mathrm{id} \otimes q^1 \\ -\mathrm{id} \otimes q^0 & p^0 \otimes \mathrm{id} \end{pmatrix}} \begin{array}{c} P^0 \otimes Q^0 \\ \oplus \\ P^1 \otimes Q^1 \end{array} \right)$$

Lemma 5.15. *The tensor product \otimes is well-defined (up to isomorphism), and moreover, the choice of where to put the factor $1/2$ is canonical in that different choices*

$$\alpha \begin{pmatrix} p^0 \otimes \mathrm{id} & -\mathrm{id} \otimes q^1 \\ \mathrm{id} \otimes q^0 & p^1 \otimes \mathrm{id} \end{pmatrix}, \quad \beta \begin{pmatrix} p^1 \otimes \mathrm{id} & \mathrm{id} \otimes q^1 \\ -\mathrm{id} \otimes q^0 & p^0 \otimes \mathrm{id} \end{pmatrix}$$

for our differentials where $2\alpha\beta = 1$ give isomorphic matrix factorizations.

Proof. The well-definition of \otimes is clear since chain homotopies h become chain homotopies on a tensor product by $h \otimes \mathrm{id}$ (see [Wu, Lem. 3.9]).

The choice of factor $1/2$ is canonical as seen in the following morphism:

$$\begin{array}{ccccc} \begin{array}{c} P^0 \otimes Q^0 \\ \oplus \\ P^1 \otimes Q^1 \end{array} & \xrightarrow{\frac{1}{2} \begin{pmatrix} p^0 \otimes \mathrm{id} & -\mathrm{id} \otimes q^1 \\ \mathrm{id} \otimes q^0 & p^1 \otimes \mathrm{id} \end{pmatrix}} & \begin{array}{c} P^1 \otimes Q^0 \\ \oplus \\ P^0 \otimes Q^1 \end{array} & \xrightarrow{\begin{pmatrix} p^1 \otimes \mathrm{id} & \mathrm{id} \otimes q^1 \\ -\mathrm{id} \otimes q^0 & p^0 \otimes \mathrm{id} \end{pmatrix}} & \begin{array}{c} P^0 \otimes Q^0 \\ \oplus \\ P^1 \otimes Q^1 \end{array} \\ \downarrow \mathrm{id} & & \downarrow \cdot 2\alpha & & \downarrow \mathrm{id} \\ \begin{array}{c} P^0 \otimes Q^0 \\ \oplus \\ P^1 \otimes Q^1 \end{array} & \xrightarrow{\alpha \begin{pmatrix} p^0 \otimes \mathrm{id} & -\mathrm{id} \otimes q^1 \\ \mathrm{id} \otimes q^0 & p^1 \otimes \mathrm{id} \end{pmatrix}} & \begin{array}{c} P^1 \otimes Q^0 \\ \oplus \\ P^0 \otimes Q^1 \end{array} & \xrightarrow{\beta \begin{pmatrix} p^1 \otimes \mathrm{id} & \mathrm{id} \otimes q^1 \\ -\mathrm{id} \otimes q^0 & p^0 \otimes \mathrm{id} \end{pmatrix}} & \begin{array}{c} P^0 \otimes Q^0 \\ \oplus \\ P^1 \otimes Q^1 \end{array} \\ \downarrow \mathrm{id} & & \downarrow \cdot \beta & & \downarrow \mathrm{id} \\ \begin{array}{c} P^0 \otimes Q^0 \\ \oplus \\ P^1 \otimes Q^1 \end{array} & \xrightarrow{\frac{1}{2} \begin{pmatrix} p^0 \otimes \mathrm{id} & -\mathrm{id} \otimes q^1 \\ \mathrm{id} \otimes q^0 & p^1 \otimes \mathrm{id} \end{pmatrix}} & \begin{array}{c} P^1 \otimes Q^0 \\ \oplus \\ P^0 \otimes Q^1 \end{array} & \xrightarrow{\begin{pmatrix} p^1 \otimes \mathrm{id} & \mathrm{id} \otimes q^1 \\ -\mathrm{id} \otimes q^0 & p^0 \otimes \mathrm{id} \end{pmatrix}} & \begin{array}{c} P^0 \otimes Q^0 \\ \oplus \\ P^1 \otimes Q^1 \end{array} \end{array}$$

which is an isomorphism. □

The following claim, however, is harder to prove:

Claim 5.16. $\Psi(\mathcal{O}_X)$ is the identity under the tensor product \otimes defined above on the image of the Picard group through Ψ .

By Theorem 2.19, it suffices to show that $\text{Cok}(\Psi(\mathcal{O}_X) \otimes Q^\bullet) \cong \text{Cok} Q^\bullet = \text{coker } q^0$, i.e.,

$$\text{coker} \begin{pmatrix} p^0 \otimes \text{id}(d) & -\text{id} \otimes q^1 \\ \text{id}(d) \otimes q^0 & p^1 \otimes \text{id} \end{pmatrix} \cong \text{coker } q^0.$$

We don't know if this would work in the general case, although we're pretty sure it does, and so we consider the following toy example:

Example 5.17. Consider the case $\Psi(\mathcal{O}_X) \otimes \Psi(\mathcal{O}_X)$ where $B = \mathbb{C}[[x]]$ and $W = x$. Recall from §3.2 that we have the following matrix factorization for $\Psi(\mathcal{O}_X)$:

$$B \begin{array}{c} \xrightarrow{-1(-1)} \\ \xleftarrow{-x} \end{array} B$$

The corresponding matrix factorization $\Psi(\mathcal{O}_X) \otimes \Psi(\mathcal{O}_X)$ is given by

$$B^2 \begin{array}{c} \xrightarrow{\frac{1}{2} \begin{pmatrix} -1 & x \\ -1 & -x \end{pmatrix}} \\ \xleftarrow{\begin{pmatrix} -x & -x \\ 1 & -1 \end{pmatrix}} \end{array} B^2$$

Note the cokernel of $-1(-1)$ is isomorphic to $k[-1]$, as also computed in §3.2 and so it suffices to show that

$$\text{coker} \begin{pmatrix} -1 & x \\ -1 & -x \end{pmatrix} \cong k[-1]$$

By row reduction, we see that

$$\text{coker} \begin{pmatrix} -1 & x \\ -1 & -x \end{pmatrix} \cong \text{coker} \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \cong k.$$

This actually suggests that the correct tensor product structure on $\text{HMF}^{\text{gr}}(W)$ is probably given by $- \otimes -[-1]$.

5.3. Possible Connections to the Grothendieck Group. We now return to our first motivating question from the introduction, in the case of the Grothendieck Group $K(\mathcal{T})$ for a triangulated category \mathcal{T} . First we define what we mean by the Grothendieck group:

Definition 5.18 ([SGA5, VIII, §2]). Let \mathcal{T} be a triangulated category. The *Grothendieck group* $K(\mathcal{T})$ of \mathcal{T} is the quotient of the free abelian group on the isomorphism classes $[A]$ of objects of \mathcal{T} by the subgroup generated by expressions $[A] - [B] + [C]$ for every distinguished triangle $A \rightarrow B \rightarrow C \rightarrow A[1]$ in \mathcal{T} .

Note this closely mirrors the definition for the abelian case (cf, [SGA5, VIII, §1], see [AM69, p. 88] for an elementary introduction to the theory).

When we have a tensor product \otimes on our category \mathcal{T} , then, we get the *Grothendieck ring* by defining $[A] \otimes [B] := [A \otimes B]$. Note that since we then have a ring and the corresponding scheme $\text{Spec}(K(\mathcal{T}))$, this prompts a comparison with Balmer's construction [Bal05]:

Question 5.19. If \mathcal{T} is a tensor triangulated category, and $K(\mathcal{T})$ its Grothendieck ring, then how are Balmer's $\text{Spec}(\mathcal{T})$ and the usual $\text{Spec}(K(\mathcal{T}))$ related? E.g., is there a morphism of locally ringed spaces between them?

In the case of §5.1, this gives rise to the analogous question:

Question 5.20. If $K(\underline{\mathbf{HMF}}^{\text{gr}}(W))$ is the Grothendieck ring of the graded tensor triangulated category $\underline{\mathbf{HMF}}^{\text{gr}}(W)$, and provided that we can make sense of $\text{Proj}(\underline{\mathbf{HMF}}^{\text{gr}}(W))$ as a modification of Balmer’s construction, then how are $\text{Proj}(\underline{\mathbf{HMF}}^{\text{gr}}(W))$ and the usual $\text{Proj}(K(\underline{\mathbf{HMF}}^{\text{gr}}(W)))$ related? Is there a morphism of locally ringed spaces between them?

Note that we are assuming here that $K(\underline{\mathbf{HMF}}^{\text{gr}}(W))$ would have a graded structure induced by the grading on $\underline{\mathbf{HMF}}^{\text{gr}}(W)$.

There are two (very) special cases in which the answer to Question 5.19 might be easy:

Example 5.21. Let $X = \mathbb{P}^1$ be the projective line over a field k . X is then a Fano variety and so has ample anticanonical sheaf; this implies the triangulated category of perfect complexes $\mathbf{D}^{\text{perf}}(X)$ is equivalent to $\mathbf{D}^b(\text{coh } X)$ [Bal02, §1]. By Balmer’s reconstruction theorem [Bal05, Thm. 6.3], we have $\text{Spec}(\mathbf{D}^{\text{perf}}(X)) \simeq X$, and so $\text{Spec}(\mathbf{D}^b(\text{coh } X)) \simeq X$.

We also have the following computation of the Grothendieck ring [Man69, Ex. 3.11]:

$$K(\mathbb{P}^1) \cong \mathbb{Z}[x]/((1-x)^2)$$

The question then reduces to a comparison between \mathbb{P}^1 and $\text{Spec}(\mathbb{Z}[x]/((1-x)^2))$.

Example 5.22. Slightly more generally, let X be a smooth curve with ample anticanonical or canonical sheaf. Again, the triangulated category of perfect complexes $\mathbf{D}^{\text{perf}}(X)$ is equivalent to $\mathbf{D}^b(\text{coh } X)$ [Bal02, §1]. Again by Balmer’s reconstruction theorem, we have $\text{Spec}(\mathbf{D}^b(\text{coh } X)) \simeq X$.

Recall from [Har77, p. 149] that the Grothendieck group $K(\mathbf{D}^b(\text{coh } X)) \cong K(\text{coh } X) \cong \text{Pic}(X) \oplus \mathbb{Z}$ for X a smooth curve. After giving $\text{Pic}(X) \oplus \mathbb{Z}$ the ring structure induced by \otimes , the question then reduces to asking how X and $\text{Spec}(\text{Pic}(X) \oplus \mathbb{Z})$ are related.

Note that in these particular situations, Orlov’s equivalence fails since we no longer have a hypersurface of requisite degree (the Gorenstein parameter is nonzero), and moreover $\mathbf{D}_{\text{Sg}}^{\text{gr}}(A)$ is trivial and so $\mathbf{HMF}(W)$ would also be trivial. Thus, this line of reasoning would not be useful for Question 5.20.

CONCLUDING REMARKS

A common thread throughout much of the current research related to matrix factorizations and Orlov’s equivalence is related to how we could possibly take advantage of the dictionary between coherent sheaves and matrix factorizations to prove theorems on one side or the other. While discussing this equivalence with other people, I have come to realize that there has not been much activity on the matrix factorization side, most probably because it is much harder to prove theorems with matrix factorizations than with coherent sheaves, which are much better understood because of the abundance of work done by Grothendieck, Serre, Hartshorne, Artin, and others.

It is my personal opinion, however, that there must be something hidden in the connection between matrix factorizations and tensor triangular geometry. The possibility outlined in §5.1 concerning a re-formulation of Balmer’s work from [Bal05] in the case of graded tensor triangular categories and the projective scheme associated with it, might suggest something similar to Balmer’s reconstruction theorem [Bal10, Thm. 63, Rem. 64]. And there probably is something that can be garnered from the affine case as well, as outlined in §5.2.

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