Fiber Bundles in Analytic, Zariski, and Étale Topologies

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a senior thesis submitted in partial fulfillment of the requirements for the degree of Bachelor of Arts in Mathematics at Princeton University

May 5, 2014

Acknowledgments

My time here at Princeton University has been quite an adventure, and it is hard to decide who to thank and acknowledge on these pages.

First of all, I would like to thank my adviser, János Kollár. It is only through his guidance and patience that I was able to pinpoint a good topic to write about in this thesis, and then complete it to the state where it is at now. No one knows better than he does how confused I can get even with the most basic parts of mathematics, and I cannot stress enough the extent to which I am indebted to his teaching. I know that the lessons and ideas I have learned from him will stay with me for years to come.

I would also like to express my sincerest thanks to my second reader for this thesis, Zsolt Patakfalvi. My experience with him in the last year working on algebraic geometry has been an enlightening one, and I especially appreciate the effort he put into making sure I understood the foundations of modern algebraic geometry, without losing track of the geometric insight and intuition at its core.

My teachers and colleagues in the mathematics department are also very deserving of recognition. I came to Princeton not sure whether or not I wanted to pursue mathematics as my undergraduate career, and without their encouragement and the conversations I had with them, I might not be writing this thesis today. They have shown me that the beauty of mathematics is really unparalleled, and that in many ways mathematics is like art or poetry, full of images and stories that must be discovered and unraveled slowly with a good cup of coffee. In particular, Kevin Tucker and Yu-Han Liu introduced me to algebraic geometry, and it is because of their enthusiasm that I focused my attention on this field in the first place.

Lastly, I would like to thank my family and friends for supporting me these last four years. My love of mathematics would never have begun without my father's explanations of how mathematics is the basis for the music my mother taught me to appreciate so much. And without my friends, especially those from VTone, I would never have been able to stay up so many nights and work so hard to complete this thesis and all the other work I've done these last four years.

This thesis represents my own work in accordance with University regulations.

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May 5, 2014
Princeton, NJ

Abstract

We compare the behavior of fiber bundles with structure group G for G the general linear group GL, the projective general linear group PGL, and the general affine group GA. We prove a criterion for when a GA-bundle is in fact a GL-bundle, i.e., a vector bundle, over the Riemann sphere $\mathbb{P}^1$. We also discuss the behavior of fiber bundles under different topologies; specifically, we compare the analytic versus Zariski topologies over the Riemann sphere $\mathbb{P}^1$, and compare the étale and Zariski topologies on schemes in general.

Contents

Introduction

Let X be a projective algebraic variety defined over the complex numbers $\mathbb{C}$, i.e., X is defined as the zero set of homogeneous polynomials $\{f_i\colon \mathbb{C}^{n+1} \to \mathbb{C}\}_{1\le i\le s}$ in the complex projective space $\mathbb{P}^n_{\mathbb{C}}$ of dimension n. X can then be given what is called the Zariski topology, generated by the open basis of sets defined by the complement in X of zero sets of some other homogeneous polynomials $\{g_j\}$. An important feature of algebraic geometry over $\mathbb{C}$ is the following: if X is smooth, i.e., for all points $x = (x_0 : \cdots : x_n) \in X$, the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_s}{\partial x_1} & \frac{\partial f_s}{\partial x_2} & \cdots & \frac{\partial f_s}{\partial x_n} \end{pmatrix}$$

has rank $n - d$, where d is the dimension of X, then X also has the structure of a complex analytic manifold, i.e., a complex manifold that is locally biholomorphic to $\mathbb{C}^d$, with a topology given by the subspace topology from $\mathbb{P}^n_{\mathbb{C}}$ with the standard topology. Call this topology the analytic topology. The geometry of projective algebraic varieties over $\mathbb{C}$, then, has the benefit of having topological, analytic, and algebraic methods at its disposal; in particular, projective algebraic varieties of dimension 1 are widely studied as Riemann surfaces.

The richness of the structure on a complex projective variety, however, also means that there are non-trivial choices to be made. Recall, for example, the notion of a vector bundle:

Definition. Let X be a projective algebraic variety over $\mathbb{C}$. A *vector bundle* is a space E together with a morphism $p\colon E \to X$ such that for any $x \in X$ there is an open set U containing x and an isomorphism $g\colon U \times \mathbb{C}^r \to p^{-1}(U)$ such that $p \circ g(u, e) = u$ for any $u \in U$, $e \in \mathbb{C}^r$, and such that if U' is another open set containing x with isomorphism $g'\colon U' \times \mathbb{C}^r \to p^{-1}(U')$, then the two vector space structures on $p^{-1}(x)$ are the same.

Note that this definition works in two different settings:

(1) Consider X with the analytic topology, with E another complex analytic manifold, and morphisms given by holomorphic functions. This gives holomorphic vector bundles over X.
(2) Consider X with the Zariski topology, with E another algebraic variety over $\mathbb{C}$, and morphisms given by regular functions, i.e., functions locally given by polynomials. This gives algebraic vector bundles over X.

At first glance, these notions give very different objects: indeed, while every algebraic vector bundle is clearly holomorphic, there is no reason to believe that the converse is the case. Indeed, the fact that on a projective variety, holomorphic and algebraic vector bundles are the same is a deep result due to Serre [GAGA, Prop. 18]. Moreover, this result is fairly special, in that replacing the vector spaces $\mathbb{C}^r$ with some other space F gives fiber bundles that are no longer always algebraic: sometimes a holomorphic fiber bundle is no longer an algebraic bundle.

Our goal is to investigate the difference between the behavior of fiber bundles in the two topologies above, and also the difference between the behavior when we change the fibers of our bundle. We use Grothendieck's theorem classifying algebraic vector bundles over $\mathbb{P}^1$ [Gro57] as a point of departure.

After proving his result, we extend our methods to some kinds of fiber bundles over $\mathbb{P}^1$, specifically those listed in Example 1.11. In particular, we look at what happens if in the definition above for a vector bundle, we assume that $p^{-1}(x)$ only has the same *affine* structure when looking at two different neighborhoods U, U' on which we have morphisms g, g'. In this situation, we prove a new criterion for when the "affine" bundle thus obtained is in fact a vector bundle.

Finally, we discuss fiber bundles in the general context of schemes. For schemes, we can introduce what is called the étale topology, which in some ways provides an analogue for analytic bundles over complex varieties. Finally, we give some comparisons between the behavior of fiber bundles in the Zariski and étale topologies.

Part 1

Algebraic and holomorphic fiber bundles over the Riemann sphere

CHAPTER 1

Fiber bundles with structure group

We begin the first half of this thesis by talking about complex algebraic varieties and complex analytic manifolds, and defining what are fiber bundles on them. Fiber bundles are a generalization of vector bundles. Recall that for a vector bundle, each fiber has the structure of a vector space. For a fiber bundle, though, we just require that each fiber is isomorphic to some fixed space F.

Unlike in French, we do not have the luxury of using the word "variété" to refer to both algebraic varieties and complex analytic manifolds; regardless, in the following we use the word "variety" without qualification to emphasize the fact that the definitions are the same in both contexts, by replacing the notion of a morphism in the right way, i.e., with regular maps in the algebraic case, and holomorphic maps in the analytic case. In the algebraic context, we do not have to work over the field $\mathbb{C}$; we will state when something depends on the choice of field k.

1. Groups and group actions on varieties

Oftentimes when we have a variety, we would like to analyze the way a group can act on it. The main examples we have in mind are the classical groups, e.g., the general linear group acting on a vector space, and subgroups thereof. In general, we have the following definition:

Definition 1.1. Let G be a variety. G is an *algebraic* or *Lie group*, depending on the context, if the underlying set of G forms a group, such that multiplication $G \times G \to G$, $(g, g') \mapsto gg'$ and inversion $G \to G$, $g \mapsto g^{-1}$ are morphisms.

Definition 1.2. Let G be an algebraic or Lie group, and X a variety. *G acts (on the left)* of X if there is a morphism $m_X : G \times X \to X$ such that $m_X(1, x) = x$ and $m_X(g'g, x) = m_X(g', F(g, x))$ for all $x \in X$, $g, g' \in G$. We write $m_X(g, x) = g \cdot x$. If m_X is understood, we call X a *G-variety*. A *morphism of G-varieties* is a morphism $\pi : X \to Y$ between G-varieties such that π is *G-invariant*, that is, $\pi(g \cdot x) = g \cdot \pi(x)$ for all $x \in X$.

Example 1.3. Let $G = \mathrm{GL}_r(k)$, the *general linear group of degree r over k*. This is given by $r \times r$ matrices $A = (a_{ij})$ with entries in k such that $\det A \neq 0$. GL_r acts on affine r-space $\mathbb{A}^r$ with a chosen basis $\{e_i\}$ by the map

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_r \end{pmatrix} \mapsto A \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_r \end{pmatrix}.$$

It is an algebraic/Lie group by taking the open subset of $\mathbb{A}^{r^2}$ defined by those $(a_{ij}) \in \mathbb{A}^{r^2}$ such that $\det(a_{ij}) \neq 0$.

We note, however, that the automorphisms defined by matrices in $\mathbb{A}^r$ are not all of the automorphisms of $\mathbb{A}^r$; for example, a translation $v \mapsto v + w$ for fixed $w \in \mathbb{A}^r$ gives an automorphism of $\mathbb{A}^r$, but is not given by a matrix in GL_r since $0 \in \mathbb{A}^r$ is not fixed by this automorphism. By embedding $\mathbb{A}^r$ in $\mathbb{P}^r$, even, there are already more automorphisms of $\mathbb{A}^r$. To describe these, we first describe the analogue of GL_r for projective r-space:

Example 1.4. Let $G = \mathrm{PGL}_r(k) = \mathrm{GL}_{r+1}(k)/k^\times$, the *projective general linear group of degree r* over k. This is given by $(r+1) \times (r+1)$ matrices A in GL_{r+1} modulo scalar multiplication by $k^\times$. PGL_r acts on projective r-space $\mathbb{P}^r$ by the map

$$\begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_r \end{bmatrix} \mapsto A \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_r \end{bmatrix}.$$

It is an algebraic/Lie group by taking the open subset of $\mathbb{P}^{r^2}$ defined by those $[a_{ij}] \in \mathbb{P}^{r^2}$ such that $\det[a_{ij}] \neq 0$.

Unlike for GL_r acting on $\mathbb{A}^r$, however, we know that PGL_r is in fact the entire group of automorphisms of $\mathbb{P}^r$:

Proposition 1.5. *Every automorphism of* PGL_r *is linear, i.e., is induced by an automorphism* $A \in \mathrm{GL}_{r+1}$ *of* $\mathbb{A}^{r+1}$.

Proof ([Har95, p. 229]). Let H and $L \subset \mathbb{P}^r$ be a hyperplane and a line in $\mathbb{P}^r$, respectively, meeting at one point. By Bézout's theorem [Har95, Thm. 18.3], since an automorphism φ of $\mathbb{P}^r$ must preserve intersection numbers, we have that H maps to a hyperplane and L to a line, such that they still meet at one point. This implies that hyperplanes must map to hyperplanes under the automorphism φ. If on an affine chart φ is given by

$$(z_1, \dots, z_r) \mapsto (w_1, \dots, w_r), \quad w_i = \frac{f_i(z_1, \dots, z_r)}{g_i(z_1, \dots, z_r)}$$

the requirement that φ maps hyperplanes to hyperplanes implies that a linear relation between the w_i implies one between the z_i, hence the degrees of the f_i, g_i are all at most 1, with all g_i scalar multiples of each other. □

Note that Bézout's theorem only works if our base field is algebraically closed; we will give a different proof later that works for non-algebraically closed fields k as well as finitely generated UFD's over k.

This example PGL_r gives us a slightly larger group of automorphisms of $\mathbb{A}^r$:

Example 1.6. Let $G = \mathrm{GA}_r(k) \subset \mathrm{PGL}_r(k)$ be the subgroup that that fixes the hyperplane $\{x_0 = 0\}$ in $\mathbb{P}^r$. This is the *general affine group of degree r* over k, and is given by $(r+1) \times (r+1)$ matrices (a_{ij}) in GL_{r+1} modulo scalar multiplication by $k^\times$ such that that $a_{01} = a_{02} = \cdots =$

$a_{0r} = 0$. Note then that $a_{00} \neq 0$, hence normalizing by a_{00}^{-1} gives that every element of GA_r has a representative of the form

$$\left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline * & & & \\ \vdots & & A & \\ * & & & \end{array}\right)$$

where $A \in \mathrm{GL}_r(k)$. GA_r is an algebraic/Lie group since the requirement $a_{01} = a_{02} = \cdots = a_{0r} = 0$ defines a closed subset of GL_{r+1} or PGL_r that is also a subgroup. Note that by taking the hyperplane $\{x_0 = 0\}$ out of the $\mathbb{P}^r$ that GA_r acts on, we get an automorphism of $\mathbb{A}^r$.

This way of representing elements in GA_r gives rise to the following diagram relating our three examples above:

$$\begin{array}{ccccc} & & & & \mathrm{GL}_{r+1}(k) \\ & & & \nearrow & \downarrow \\ \mathrm{GL}_r(k) & \hookrightarrow & \mathrm{GA}_r(k) & \hookrightarrow & \mathrm{PGL}_r(k) \end{array} \tag{1.7}$$

Note this relationship is on the level of algebraic/Lie groups; there is no reason why we should expect *morphisms* to lift from PGL_r to GL_r, for example. This is the main topic of Chapter 2.

2. Definition of fiber bundles

We can now define our main object of study, that is, fiber bundles with structure group. These generalize vector bundles, and provide a good framework in which to talk about vector bundles and related constructions with different fibers. For example, vector bundles have fibers that are isomorphic to some vector space V, but we can always stipulate that the fibers are isomorphic to some other variety or scheme F. In the examples we are interested in, moreover, there is some group G acting on these fibers F, just like how GL acts on the fibers of a vector bundle. Let G be an algebraic or Lie group. For more discussion, see [Gro58a].

DEFINITION 1.8. A *fiber system with structure group G and fiber F* is a G-variety E together with a morphism $\pi: E \to X$ such that $\pi^{-1}(x) \cong F$ for some fixed G-variety F, and $G \cdot \pi^{-1}(x) \subset \pi^{-1}(x)$ for all $x \in X$. A *morphism* of two fiber systems $\pi: E \to X$ and $\pi': E' \to X$ is a morphism of G-varieties $f: E \to E'$, such that it makes the diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & X & \end{array}$$

commute. If $f: X' \to X$ is a morphism, we define the *pullback* fiber system f^*E as the G-variety $E' = X' \times_X E$ together with the morphism $\pi': E' \to X'$ in the bottom fiber square below, and the G-action is given by the top square:

$$\begin{array}{ccc} G \times E' & \longrightarrow & G \times E \\ {\scriptstyle m_{E'}}\downarrow & & \downarrow{\scriptstyle m_E} \\ E' & \longrightarrow & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{f} & X \end{array}$$

In the special case $X' \subset X$, we call $f|_{X'} := f^*E$ the *restriction* of the fiber system to X'.

It is hard to automatically see which fiber systems are isomorphic, that is, when there are two morphisms f, f^{-1} whose composition in both directions is the identity, so we have the following criterion:

Lemma 1.9. *Let $\pi : E \to X$ and $\pi' : E' \to X$ be two fiber systems and let $f : E \to E'$ be a morphism between them. f is an isomorphism if and only if for all $x \in X$, there is a neighborhood U_x containing x such that f induces an isomorphism $E|_{U_x} \xrightarrow{\sim} E'|_{U_x}$.*

Proof. This follows since morphisms of G-varieties are defined locally. ☐

We would moreover like our fiber systems to have a "local triviality condition":

Definition 1.10. The *trivial fiber system* is that defined by the projection $X \times F \to X$. A *fiber bundle with structure group G and fiber F* or simply a *G-bundle with fiber F* is a fiber system over X together with an open cover $\{U_i\}$ of X and isomorphisms $\varphi_i : \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times F$,

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow[\sim]{\varphi_i} & U_i \times F \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & U_i & \end{array}$$

commutes, and such that the maps

$$\varphi_i \circ \varphi_j^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$$

are given by $(x, \xi) \mapsto (x, t_{ij}(x)\xi)$ for some morphisms $t_{ij} : U_i \cap U_j \to G$, which satisfy the cocycle condition $t_{ik} = t_{ij}t_{jk}$. Morphisms of fiber bundles are morphisms as fiber systems.

Examples 1.11. In this thesis, we focus on the following examples:

(1) Let $G = \mathrm{GL}_r$ acting on $F = \mathbb{A}^r$ (1.3). F is then a vector space by choosing the origin to be the stabilizer of the G-action, and the GL_r-bundles with fiber $\mathbb{A}^r$ are called *vector bundles* on X.
(2) Let $G = \mathrm{PGL}_r$ acting on $F = \mathbb{P}^r$ (1.4). The PGL_r-bundles with fiber $\mathbb{P}^r$ are called *projective bundles* on X.
(3) Let $G = \mathrm{GA}_r$ acting on $F = \mathbb{A}^r$ (1.6). The GA_r-bundles with fiber $\mathbb{A}^r$ are called *affine bundles* on X. Note they are a special case of projective bundles, and that vector bundles are a special case of affine bundles.

These fiber bundles are related in the same way as the groups GL, PGL, GA are as illustrated in (1.7). The main goal of the first half of this thesis is to describe the relationship between these different fiber bundles over the base space $X = \mathbb{P}^1$.

Remark 1.12. Note that for vector bundles, the transition data t_{ij} can be viewed as $r \times r$ matrices with entries that are analytic functions on $U_i \cap U_j$ in the analytic case, or regular functions in $\mathcal{O}(U_i \cap U_j)$ in the algebraic case. We also remark here about how different choices of t_{ij} can give isomorphic vector bundles. Suppose E is given by matrices t_{ij} and E' by t'_{ij}. Then, suppose t'_{ij} can be written $t'_{ij} = f_i t_{ij}$ for some f_i a matrix of analytic (resp. regular) functions on U_i. The morphism $E \to E'$ then defined by gluing together the morphisms $\varphi_i'^{-1} \circ f_i \circ \varphi_i$ is an isomorphism, hence $E \cong E'$ as vector bundles.

A key question we ask is whether or not all projective bundles arise from projectivizing vector bundles, i.e., by taking the quotient by $k^\times$ in each fiber. If this were the case, then we could always lift our transition matrices in PGL to GL and work with transition matrices only. This is the basis for why our methods in Chapter 2 work, where we can compute everything in terms of matrices.

Remark 1.13. For a general fiber bundle (without a structure group), we can define a fiber bundle as a fiber space $E \to X$ that is locally isomorphic to a trivial bundle, i.e., such that for all $x \in X$ there is a neighborhood U_x such that $E|_{U_x}$ is isomorphic to the trivial system on E. In this case, the isomorphisms $X \times F \to X \times F$ as fiber systems can be identified with maps $f : X \to \mathrm{Aut}(F)$ [Gro58a, Prop. 1.4.1]. We would like a similar statement for fiber systems with a structure group that are locally isomorphic to a trivial bundle in the above sense, so that we can identify isomorphisms with maps $X \to G$. This is an issue, however, since while we do get a map $f : X \to \mathrm{Aut}_G(F)$, the fact that G may not act freely and transitively on F means that we cannot assume that this map becomes a morphism $X \to G$, for there might not be an element $g \in G$ such that $f(x) = g \cdot -$, or even if there is one, it may not be unique. For example, suppose $U_1, U_2 \subsetneq X$ form an open cover of X such that $U_1 \cap U_2 \neq \emptyset$. We can have the group $G = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ act on $\mathbb{A}^1$ by $(i,j) \cdot v \mapsto i \cdot v$, where $i, j \in \{+1, -1\}$. Then gluing together $U_i \times \mathbb{A}^1$ via the map $(x, v) \mapsto (x, cv)$ for $c \in k^\times - \{\pm 1\}$ gives a fiber system with structure group G, but this is not a fiber bundle for the transition function is not in G. Moreover, even if $c = 1$, $(1, \pm 1) \cdot v = v$, so there is ambiguity as to which element in G to choose to define our fiber bundle. This is why it is often more convenient to work with *principal bundles,* which are fiber bundles with fibers isomorphic to G, since in this case we are guaranteed that G acts freely and transitively on fibers. This is the viewpoint we will take in Chapter 4.

3. A comparison between the analytic and Zariski topologies

We will see later that for $X = \mathbb{P}^1$, the choice of analytic versus Zariski topology on X does not matter when it comes to what fiber bundles we can have, at least for our examples in Example 1.11. On other base spaces X, though, it can make a difference. This is illustrated by the following example:

Example 1.14. Let $X = \mathbb{A}^2 - \{st = 0\}$, the affine plane minus the two lines $\{s = 0\}$ and $\{t = 0\}$, and define the fiber system E by having fibers $\{x^2 + sy^2 = tz^2\} \subset \mathbb{P}^2$ projecting onto

$(s,t) \in X$. This is a fiber system with structure group PGL_1 since for fixed s,t, $\{x^2+sy^2 = tz^2\}$ is a smooth conic, hence is isomorphic to $\mathbb{P}^1$.

Now note $X = \mathbb{C}^\times \times \mathbb{C}^\times$. Giving X the analytic topology, denote $\mathbb{C}_\pm$ to be $\mathbb{C}$ minus the $\pm$ part of the $\mathbb{R}$ axis, and let

$$U_1 = \mathbb{C}_+ \times \mathbb{C}_+, \quad U_2 = \mathbb{C}_+ \times \mathbb{C}_-, \quad U_3 = \mathbb{C}_- \times \mathbb{C}_+, \quad U_4 = \mathbb{C}_- \times \mathbb{C}_-.$$

On each of these open sets, we have the trivialization given by

$$\begin{aligned} U_i \times \mathbb{P}^1 &\to E|_{U_i} \\ (s,t) \times (u:v) &\mapsto \left(u^2 - v^2 : \frac{2uv}{s^{1/2}} : \frac{u^2+v^2}{t^{1/2}}\right) \end{aligned}$$

where $s^{1/2}, t^{1/2}$ are the positive or negative square roots depending on the U_i. The transition data is then given by

$$t_{13} = t_{24} = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \quad t_{12} = t_{34} = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, \quad t_{14} = t_{23} = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \tag{1.15}$$

and so $E \to X$ defines a holomorphic projective bundle.

On the other hand, in the Zariski topology, this cannot be locally trivial since the fiber at the generic point is the scheme $\{x^2+sy^2 = tz^2\} \to \operatorname{Spec} \mathbb{C}(s,t)$, which has no rational points. For suppose it had a point (x_0,y_0,z_0) where $x_0,y_0,z_0 \in \mathbb{C}(s,t)$. Clearing denominators we can assume $x_0,y_0,z_0 \in \mathbb{C}[s,t]$. Let $s^{a_1}t^{b_1}, s^{a_2}t^{b_2}, s^{a_3}t^{b_3}$ be the terms with lowest total degree in x_0,y_0,z_0 respectively. Then, the term with lowest total degree in $x^2+sy^2-tz^2$ is one of $s^{2a_1}t^{2b_1}, s^{2a_2+1}t^{2b_2}, s^{2a_3}t^{2b_3+1}$, but these clearly do not cancel.

Note, however, that the matrices in (1.15) *do* define a projective bundle in the Zariski topology, which shows that there can exist fiber systems that are isomorphic by holomorphic maps, but not algebraic maps.

We cannot hope to construct a similar example with plane conics as fibers with a base space $\mathbb{C}$ because of the following:

Theorem 1.16 (Tsen [Tse33]). *Let $K = k(s)$, k algebraically closed. Then any homogeneous polynomial $f(t_1,\dots,t_n)$ in K of degree $d < n$ has a non-trivial zero.*

Proof. Let $f(t_1,\dots,t_n) = \sum a_{i_1\cdots i_n} t_1^{i_1}\cdots t_n^{i_n}$ is a homogeneous polynomial of degree $d<n$ with coefficients in K. Clearing denominators, we can assume that the coefficients are in $k[s]$. Now let $\delta = \sup \deg(a_{i_1\cdots i_n})$. We would like to find a solution with each t_i having degree at most N in s. The equation $f = 0$ defines a homogeneous system of equations in the $n\times(N+1)$ coefficients of the polynomials $t_i(s)$ by replacing each t_i with $t_i(s)$. Since f has degree at most $\delta + Nd$ in s, we then have $\delta + Nd + 1$ equations in $n\times(N+1)$ variables. Since k is algebraically closed, we then have a nontrivial solution if $n(N+1) > Nd+\delta+1$, which occurs for large enough N since $d<n$. □

So if we had a family E of non-degenerate conics over a Zariski-open subset of $\mathbb{A}^1$, our argument above no longer works as the generic fiber is has a point by Tsen's theorem, hence is rational.

Remark 1.17. At first glance, the matrices in (1.15) look like they might not commute if lifted up to GL_2. We will return to this example in Remark 2.17; we remark for now that surprisingly, the bundle defined by these matrices *does* lift to a vector bundle, even if algebraically, the fibration of conics that we defined does not.

CHAPTER 2

Fiber bundles on the Riemann sphere

Analyzing the examples of fiber bundles on the Riemann sphere $\mathbb{P}^1$ we are interested in is convenient because, as we shall see, we can do so using matrices and linear algebra.

1. The case of vector bundles

In this section, we classify vector bundles on $\mathbb{P}^1$. Recall that $\mathbb{P}^1$ is defined as follows: let $U_0 = \operatorname{Spec} k[s]$ and $U_1 = \operatorname{Spec} k[t]$, $U_{01} = \operatorname{Spec} k[s, s^{-1}] = U_0 - \{0\}$ and $U_{10} = \operatorname{Spec} k[t, t^{-1}] = U_1 - \{0\}$. $\mathbb{P}^1$ is then obtained by gluing together U_0, U_1 along the isomorphism identifying U_{01} and U_{10} by the isomorphism induced by $k[s, s^{-1}] \xrightarrow{\sim} k[t, t^{-1}]$ defined by $s \mapsto t^{-1}$. Note that each U_i is a copy of $\mathbb{A}^1$. Similarly, we can get $\mathbb{P}^1$ as a complex analytic manifold by doing the same. Note that the proofs in this section do not depend on our field k.

Our goal is to prove the following result due to Grothendieck:

Theorem 2.1 ([Gro57, Thm. 2.1]). *Let E be a rank r vector bundle over $\mathbb{P}^1$. Then E is isomorphic to a direct sum of line bundles*

$$E \cong \mathcal{O}(\kappa_1) \oplus \cdots \oplus \mathcal{O}(\kappa_r), \quad \kappa_1 \geq \cdots \geq \kappa_r, \quad \kappa_i \in \mathbb{Z}, \ i = 1, \dots, r,$$

and the κ_i are uniquely determined by the isomorphism class of E.

where we note that each $\mathcal{O}(\kappa_i)$ is defined by the transition matrix $(s^{-\kappa_i})$, and that the direct sum above corresponds to taking direct sums of transition matrices.

Our strategy is as follows: we show that any vector bundle is trivial over U_0 and U_1. Then, the question reduces to how two trivial bundles on U_0, U_1 above can glue together on their intersection. The gluing operation corresponds to one transition matrix by Remark 1.12. Finally, we can obtain a diagonal transition matrix realizing our vector bundle as a direct sum of line bundles via an elementary linear algebra argument from [HM82].

We pause here for a moment to discuss the history of this result. Our approach here shows that Grothendieck's result boils down to an algebraic statement about transition matrices. However, this result on matrices was apparently known to Dedekind and Weber; see [Sch05] for a discussion. In addition, Grothendieck's theorem above in the analytic setting is also the consequence of a theorem due to G.D. Birkhoff on matrices of analytic functions [Bir13] as pointed out by Seshadri [Ses57].

To show that any vector bundle over U_0 or U_1 is trivial, it suffices to show

Proposition 2.2. *Every vector bundle on $\mathbb{A}^1$ is trivial.*

In the algebraic setting, this result is actually a rather easy consequence of the following linear algebraic result:

THEOREM 2.3 (Smith normal form [Jac85, Thm. 3.8]). *Let D be a PID, and A an $r \times r$ matrix in D. Then, there exist $P, Q \in \mathrm{GL}_r(D)$ such that*

$$PAQ = \begin{pmatrix} d_1 & & & & \\ & d_2 & & 0 & \\ & & \ddots & & \\ & & & d_p & \\ & 0 & & 0 & \\ & & & & \ddots \end{pmatrix}$$

where $d_i \neq 0$ and $d_i \mid d_j$ if $i \leq j$.

We can now prove our proposition.

PROOF OF PROPOSITION 2.2. Suppose a vector bundle $\pi : E \to \mathbb{A}^1$ is trivial on $D(f_1)$ and $D(f_2)$. We claim that it is isomorphic to the trivial bundle on $D(f_1) \cup D(f_2)$. There is a transition matrix $t_{12} \in \mathrm{GL}_r(\mathcal{O}(D(f_1 f_2))) = \mathrm{GL}_r(k[s, f_1^{-1}, f_2^{-1}])$ that defines the gluing of these trivial bundles by definition. Multiplying t_{12} by $(f_1 f_2)^e$, we have that $(f_1 f_2)^e t_{12}$ is a matrix in $k[s]$, hence has a Smith normal form; the matrices P, Q above correspond to automorphisms of the trivial bundles on $D(f_1), D(f_2)$ respectively by Remark 1.12. Note that since t_{12} is invertible in $\mathrm{GL}_r(\mathcal{O}(D(f_1 f_2)))$, its Smith normal form is of the form

$$(f_1 f_2)^e \cdot P t_{12} Q = \begin{pmatrix} f_1^{m_1} f_2^{n_1} & & & \\ & f_1^{m_2} f_2^{n_2} & & \\ & & \ddots & \\ & & & f_1^{m_r} f_2^{n_r} \end{pmatrix}$$

for $0 \leq m_1 \leq \cdots \leq m_r$ and $0 \leq n_1 \leq \cdots \leq n_r$. Then,

$$\begin{pmatrix} f_1^{e-m_1} & & & \\ & f_1^{e-m_2} & & \\ & & \ddots & \\ & & & f_1^{e-m_r} \end{pmatrix} P t_{12} Q \begin{pmatrix} f_2^{e-n_1} & & & \\ & f_2^{e-n_2} & & \\ & & \ddots & \\ & & & f_2^{e-n_r} \end{pmatrix} = \mathrm{Id}_r,$$

and so π is trivial on $D(f_1) \cup D(f_2)$. Thus, any vector bundle is trivial on $\mathbb{A}^1$, by choosing a finite open cover of $\mathbb{A}^1$ on which the bundle is trivial repeating the process above. □

We can now prove Grothendieck's theorem 2.1. In the algebraic case, this follows by a straightforward linear algebraic argument as in [HM82].

On $\mathbb{P}^1$, any vector bundle $\pi : E \to \mathbb{P}^1$, by Proposition 2.2, is trivial on U_0, U_1, hence E is the result of gluing together $U_0 \times \mathbb{A}^r$ and $U_1 \times \mathbb{A}^r$ along an isomorphism $U_0 - \{0\} \times \mathbb{A}^r \xrightarrow{\sim} U_1 - \{0\} \times \mathbb{A}^r$ which is given by $(s, v) \mapsto (s^{-1}, A(s, s^{-1})v)$, where $A(s, s^{-1})$ is a $r \times r$ matrix with entries that are regular functions.

Let $\pi : P \to \mathbb{P}^1$ be a rank r vector bundle. The matrix $A(s, s^{-1})$ mentioned above is invertible for all $s \neq 0, s^{-1} \neq 0$ since it is invertible on $U_0 \cap U_1$, hence

$$\det A(s, s^{-1}) = c \cdot s^n, \quad n \in \mathbb{Z},\ c \in k^\times. \tag{2.4}$$

By Remark 1.12, we note that depending on our choice of isomorphisms $\varphi_i : \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{A}^r$, we can have different matrices $A(s, s^{-1})$ that give isomorphic vector bundles. But isomorphisms of $U_0 \times \mathbb{A}^r$ are of the form $(s, v) \mapsto (s, U(s)v)$ for $U(s) \in \mathrm{GL}(r, k[s])$, so

$\det U(s) \in k^\times$; similarly, isomorphisms of $U_1 \times \mathbb{A}^r$ correspond to $V(s^{-1}) \in \mathrm{GL}(r, k[s^{-1}])$, so $\det V(s^{-1}) \in k^\times$. This gives us the following:

Proposition 2.5. *Isomorphism classes of rank r vector bundles over $\mathbb{P}^1$ are in one-to-one correspondence to the set*

$$\left\{ \mathrm{GL}(r, k[s, s^{-1}]) \big/ A(s, s^{-1}) \sim A'(s, s^{-1}) \iff A'(s, s^{-1}) = V(s^{-1})A(s, s^{-1})U(s) \right\}$$

where $U(s) \in \mathrm{GL}(r, k[s])$, $V(s^{-1}) \in \mathrm{GL}(r, k[s^{-1}])$.

In particular, this means that we can assume that in (2.4), the constant $c = 1$.

We now find a canonical form for matrices in this set $\mathrm{GL}(r, k[s, s^{-1}])/\sim$ to find all isomorphism classes of rank r vector bundles over $\mathbb{P}^1$.

Proposition 2.6. *Let $A(s, s^{-1}) \in \mathrm{GL}(r, k[s, s^{-1}])$ with $\det A(s, s^{-1}) = s^n$ for $n \in \mathbb{Z}$. Then, there exist $U(s) \in \mathrm{GL}(r, k[s])$, $V(s^{-1}) \in \mathrm{GL}(r, k[s^{-1}])$ such that*

$$V(s^{-1})A(s, s^{-1})U(s) = \begin{pmatrix} s^{d_1} & & & 0 \\ & s^{d_2} & & \\ & & \ddots & \\ 0 & & & s^{d_r} \end{pmatrix} \tag{2.7}$$

with $d_1 \geq d_2 \geq \cdots \geq d_r$, $d_i \in \mathbb{Z}$. The d_i are uniquely determined by $A(s, s^{-1})$.

Proof. We first prove uniqueness. Write $D(d_1, \dots, d_r)$ for the matrix on the right side of (2.7). If there are two matrices $D(d_1, \dots, d_r)$ and $D(d_1', \dots, d_r')$ equivalent to $A(s, s^{-1})$, then there are $U(s) \in \mathrm{GL}(r, k[s])$, $V(s^{-1}) \in \mathrm{GL}(r, k[s^{-1}])$ such that

$$C = V(s^{-1})D(d_1, \dots, d_r) = D(d_1', \dots, d_r')U(s). \tag{2.8}$$

We now recall the Cauchy-Binet formula [Gan59, I, §2.5]. If A is an $m \times n$ matrix and B is an $n \times q$ matrix, denoting

$$A \begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix}$$

to be the minor of A obtained by taking the determinant of A after removing all rows with index in $\{1, \dots, r\} - \{i_1, \dots, i_k\}$ and removing all columns with index in $\{1, \dots, r\} - \{j_1, \dots, j_k\}$, then

$$(AB) \begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix} = \sum_{\ell_1 < \ell_2 < \cdots < \ell_k} A \begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ \ell_1 & \ell_2 & \cdots & \ell_k \end{pmatrix} B \begin{pmatrix} \ell_1 & \ell_2 & \cdots & \ell_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix}.$$

We want to apply this to (2.8). We first note

$$D(d_1, \dots, d_r) \begin{pmatrix} \ell_1 & \ell_2 & \cdots & \ell_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix} \neq 0 \iff \ell_i = j_i \forall i.$$

Thus, we have

$$\begin{aligned} C \begin{pmatrix} 1 & 2 & \cdots & k \\ i_1 & i_2 & \cdots & i_k \end{pmatrix} &= V(s^{-1}) \begin{pmatrix} 1 & 2 & \cdots & k \\ i_1 & i_2 & \cdots & i_k \end{pmatrix} s^{d_{i_1} + d_{i_2} + \cdots + d_{i_k}} \\ &= s^{d_1' + d_2' + \cdots + d_k'} U(s) \begin{pmatrix} 1 & 2 & \cdots & k \\ i_1 & i_2 & \cdots & i_k \end{pmatrix} \end{aligned}$$

for all k and sequences $i_1 < i_2 < \cdots < i_k$. Since $\det U(s) \neq 0$, for all k, there exists at least one sequence $i_1 < i_2 < \cdots < i_k$ such that

$$U(s)\begin{pmatrix} 1 & 2 & \cdots & k \\ i_1 & i_2 & \cdots & i_k \end{pmatrix} \neq 0.$$

Thus, $d'_1 + d'_2 + \cdots d'_k \leq d_{i_1} + d_{i_2} + \cdots + d_{i_k} \leq d_1 + d_2 + \cdots d_k$ for all k. Multiplying on the right by $U(s)^{-1}$ and on the left by $V(s^{-1})^{-1}$ in (2.8) and applying the same argument, we get $d_1 + d_2 + \cdots d_k \leq d'_1 + d'_2 + \cdots d'_k$ for all k. Thus, $d_i = d'_i$ for all $i = 1, \dots, r$.

We now prove existence. We proceed by induction. For $r = 1$, the proposition clearly holds. Now for arbitrary r, we assume it works for $(r-1) \times (r-1)$ matrices. First multiply $A(s, s^{-1})$ by s^n for some $n \in \mathbb{Z}_{\geq 0}$ so that we obtain a polynomial matrix $B(s)$. Now by multiplying $B(s)$ by suitable $U(s)$ on the right, i.e., by performing elementary column operations, we can find a $B = (b'_{ij})$ with $b'_{11} \neq 0$ and $b'_{1i} = 0$ for all $i = 2, \dots, r$ (then, $b'_{11} = \gcd\{b_{1i}\}$). Then, $b'_{11} = s^{k_1}$ for some k_1 since $\det B(s)$ is some power of s. Denoting $B_2(s)$ to be the lower-right $(r-1) \times (r-1)$ submatrix of B, by induction there exist $U_2(s), V_2(s^{-1})$ such that $V_2(s^{-1})B_2(s)U_2(s)$ is of the form on the right hand side of (2.7). Then, we have

$$C(s) = \begin{pmatrix} 1 & 0 \\ 0 & V_2(s^{-1}) \end{pmatrix} B \begin{pmatrix} 1 & 0 \\ 0 & U_2(s) \end{pmatrix} = \begin{pmatrix} s^{k_1} & 0 & \cdots & 0 \\ c_2 & s^{k_2} & & 0 \\ \vdots & & \ddots & \\ c_r & 0 & & s^{k_r} \end{pmatrix} \tag{2.9}$$

for some $k_1, \dots, k_r \in \mathbb{Z}_{\geq 0}$ and $c_2, \dots, c_r \in k[s, s^{-1}]$. By multiplying $C(s)$ by suitable $V(s^{-1})$ on the left, i.e., by performing elementary row operations, we can assume that $c_i \in k[s]$ for all i.

Now consider all matrices equivalent to $B(s)$ of the form (2.9). There is one representative with k_1 maximal, for $k_2, \dots, k_r \geq 0$ implies $k_1 \leq \deg(\det B(s))$. We claim that $k_1 \geq k_i$ for all $i = 2, \dots, r$. So suppose $k_1 < k_i$. Subtracting a suitable $k[s^{-1}]$-multiple of the first row, (2.9) then has $c_i = s^{k_1+1}c'(s)$ for some $c'(s) \in k[s]$. Interchanging the first and i-th rows, we get a polynomial matrix $B'(s)$ such that the greatest common divisor of the first row is $s^{k'_1}$ with $k'_1 \geq k_1 + 1$. But applying to $B'(s)$ the same process as above for $B(s)$, we get a $C'(s)$ of the form (2.9) with $k'_1 > k_1$, contradicting maximality of k_1.

We can therefore assume in (2.9) that $k_1 \geq k_i$ and $c_i \in k[s]$ for $i = 2, \dots, r$. Now subtracting suitable $k[s]$-multiples of the ith column for $i = 2, \dots, r$ from the first column (i.e., multiplying by suitable $U(s)$ on the right) we get a matrix (2.9) with $\deg c_i \leq k_i$ for all i. Then, $\deg c_i < k_1$, and so subtracting suitable $k[s^{-1}]$-multiples of the first row from the ith row (i.e., multiplying by suitable $V(s^{-1})$ on the left), we get $c_2 = c_3 = \cdots = c_r = 0$ in (2.9). This shows that there are $k_1, \dots, k_r \in \mathbb{Z}_{\geq 0}$, $k_1 \geq k_2 \geq \cdots \geq k_r$ and $U(s) \in \mathrm{GL}(r, k[s])$, $V(s^{-1}) \in \mathrm{GL}(r, k[s^{-1}])$ such that

$$V(s^{-1})s^n A(s, s^{-1})U(s) = V(s^{-1})B(s)U(s) = D(k_1, \dots, k_r).$$

Multiplying by s^{-n} gives

$$V(s^{-1})A(s, s^{-1})U(s) = D(d_1, \dots, d_r), \quad d_i = k_i - n. \qquad \square$$

Finally, let $\mathscr{O}(d)$, $d \in \mathbb{Z}$ be the line bundle over $\mathbb{P}^1$ defined by the gluing matrix $A(s, s^{-1}) = (s^{-d})$. Then, the bundle defined by the gluing matrix $A(s, s^{-1}) = D(d_1, \dots, d_r)$ is equal to the direct sum $\mathscr{O}(-d_1), \dots, \mathscr{O}(-d_r)$, for the direct sum of two vector bundles defined by gluing matrices t_{ij} and t'_{ij} on the same cover is defined as the vector bundle defined by gluing matrices $t_{ij} \oplus t'_{ij}$. This gives us Theorem 2.1.

We pause here to give some references as to what happens in the holomorphic case. In terms of steps, the proof is the same. The fact that vector bundles over the complex plane $\mathbb{C}^1$ are trivial follows from the fact that vector bundles on non-compact Riemann surfaces are holomorphically trivial [For91, §30], or the fact that $\mathbb{C}^n$ is a Stein manifold [Gri74, VIII]. Then, the question again becomes how trivial bundles on each copy of $\mathbb{C}^1$ glue together. The critical step is the following lemma due to Birkhoff:

LEMMA 2.10 ([Bir13; Ses57, Lem. 3.31]). *Let $(l_{ij}(z))$ be a matrix of functions on $\mathbb{C}^1$ holomorphic for $|z| \geq R$ for some R, such that $\det(l_{ij}(z)) \neq 0$ for $|z| \geq R$. Then, there exists a matrix of entire functions $(\epsilon_{ij}(z))$ with $\det(\epsilon_{ij}(z)) \neq 0$ such that*

$$(l_{ij}(z))(\epsilon_{ij}(z)) = (a_{ij}(z))(\delta_{ij} z^{k_i})$$

where the k_i are integers, $(a_{ij}(z))$ is a matrix of functions holomorphic in a neighborhood of ∞ and $(\delta_{ij} z^{k_i})$ is a diagonal matrix with diagonal entries z^{k_i}.

The proof then proceeds as follows: if U_0 is the affine chart containing ∞, and the matrix $(l_{ij}(z))$ corresponds to our transition matrix for the vector bundle E, then for some small neighborhood $V \subset U_0 \cap U_1$ containing ∞ there are matrices $(\epsilon_{ij}(z))$ and $(a_{ij}(z))$ that correspond to automorphisms of E restricted to U_1 and V, respectively. The diagonal matrix $\delta_{ij} z^{k_i}$ is exactly that which we obtained algebraically above. The main difficulty of the proof of Lemma 2.10 is that unlike in the algebraic case, the isolated singularities of the entries in the matrix $l_{ij}(z)$ can be essential singularities; this is the main difficulty in Birkhoff's proof of this lemma. After reducing to the case of poles by using a Fredholm integral equation, the rest of his proof follows analgously to our own proof of Proposition 2.6.

2. The case of projective and affine bundles

We now want to extend some our results for §1 to the case of projective and affine bundles. We also want to know when an affine bundle is in fact a vector bundle; we prove such a criterion below.

2.1. Projective bundles. We first want to have an analogue of Proposition 2.2 for projective bundles. At first glance, this seems rather easy since we could just try lifting our transition matrix on some open set U from PGL_r to GL_{r+1}. The issue is that while the matrices themselves in $\mathrm{PGL}_r(k)$ lift to $\mathrm{GL}_{r+1}(k)$ as illustrated in (1.7), this does not ensure that our *maps* from U to $\mathrm{PGL}_{r+1}(k)$ and $\mathrm{GL}_{r+1}(k)$ necessarily lift as we mentioned before.

So note that after shrinking U if necessary, $U = \operatorname{Spec} R$ for some localization R of the polynomial ring $k[s]$. Then, the map $U \to \mathrm{PGL}_{r+1}$ corresponds to an automorphism of the bundle $U \times \mathbb{P}^r_k$; this is therefore an automorphism of $\mathbb{P}^r_R$ over $\operatorname{Spec} R$, that is, an automorphism of r-dimensional projective space over the ring R. We first show the following:

PROPOSITION 2.11. *If R is a finitely-generated k-algebra that is also a UFD, then all automorphisms of $\mathbb{P}^r_R$ are induced by a matrix in* $\mathrm{GL}_{r+1}(R)$.

PROOF. First consider $\mathbb{P}^r_k$; this is Proposition 1.5 but we provide another proof as promised that works over any k. Let f be an automorphism. This induces an isomorphism on Picard groups, hence $f^*\mathscr{O}(1) \cong \mathscr{O}(-1)$ or $\mathscr{O}(1)$. But a section of $\mathscr{O}(1)$ gives a section of $f^*\mathscr{O}(1)$, hence $f^*\mathscr{O}(1) \cong \mathscr{O}(1)$. Now this gives an isomorphism of vector spaces $\Gamma(\mathbb{P}^r_k, \mathscr{O}(1)) \cong \Gamma(\mathbb{P}^r_k, f^*\mathscr{O}(1))$, which corresponds to an invertible matrix in $\mathrm{GL}_{r+1}(k)$.

The case when R is a UFD that is finitely-generated as a k-algebra follows similarly by [Har77, II, Exc. 6.1], since then $\mathrm{Pic}\,\mathbb{P}^r_R \cong \mathrm{Pic}\,\mathrm{Spec}\,R \times \mathrm{Pic}\,\mathbb{P}^r_k$, and also since $\mathrm{Pic}\,\mathrm{Spec}\,R = 1$ by [Har77, II, Prop. 6.2]. We therefore have an isomorphism of free modules $\Gamma(\mathbb{P}^r_R, \mathscr{O}(1)) \cong \Gamma(\mathbb{P}^r_R, f^*\mathscr{O}(1))$, which is also given by an invertible matrix in $\mathrm{GL}_{r+1}(R)$. □

REMARK 2.12. It is important that R here is a UFD, since if we replace $\mathrm{Spec}\,R$ with any connected scheme S (e.g., if R were just a domain and not necessarily a UFD), then the group of automorphisms of $\mathbb{P}^r_S$ is isomorphic to the group of invertible sheaves $\mathscr{L}$ on S plus $(r+1) \times (r+1)$ matrices (a_{ij}) of sections of $\mathscr{L}$ such that $\det(a_{ij})$ as a section of $\mathscr{L}^{n+1}$ never vanishes; cf. [GIT, Ch. 0, §5; EGAII, §4.2]. This recovers our result since $S = \mathrm{Spec}\,R$ for R a UFD which is finitely-generated as a k-algebra has only one (isomorphism class of) invertible sheaves, namely, the structure sheaf, and so the matrix of sections given above is exactly a matrix in $\mathrm{PGL}_r(R)$.

We now want to prove the analogue of Proposition 2.2. We first prove an easy commutative algebraic result:

LEMMA 2.13. *If R is a noetherian UFD, then the localization $S^{-1}R$ is also a UFD if $0 \notin S$.*

PROOF. $S^{-1}R$ is a noetherian domain, and so by [Mat70, Thm. 47], $S^{-1}R$ is a UFD if and only if every height one prime is principal. But if P is a height one prime in $S^{-1}R$, then there is a prime ideal $Q \subset R$ such that $P = S^{-1}Q$. But localization does not affect height, hence Q has height one, hence is principal since R is a UFD, so let $Q = (a)$. Then $P = S^{-1}(a) = (\frac{a}{1})$, so P is principal, and $S^{-1}R$ is a UFD. □

We can then prove

PROPOSITION 2.14. *Every projective bundle on $\mathbb{A}^1$ is trivial.*

PROOF. Suppose our projective bundle is trivial on $D(f_1)$ and $D(f_2)$; we first show it is trivial on $D(f_1) \cup D(f_2)$. By Proposition 2.11, we have a transition matrix in $\mathrm{PGL}_r(k[s, f_1^{-1}, f_2^{-1}])$, since the localization of a UFD is a UFD by Lemma 2.13. Since there is only one transition matrix, we can lift this matrix to GL_{r+1} to define a vector bundle on $D(f_1) \cup D(f_2)$. By Proposition 2.2, this is the trivial vector bundle, and the automorphisms of the vector bundle restricted to $D(f_1)$ and $D(f_2)$ descend to automorphisms of the projective bundle restricted to $D(f_1)$ and $D(f_2)$. □

A similar argument also works for the proof of Proposition 2.6, i.e., our transition matrix in $\mathrm{PGL}(k[s, s^{-1}])$ can be lifted to one in $\mathrm{GL}(k[s, s^{-1}])$, and then we proceed as before. We moreover have the following:

Corollary 2.15. *Every projective bundle of rank r on $\mathbb{P}^1$ is of the form $P(E)$, i.e., is obtained by projectivizing each fiber in some vector bundle E.*

Proof. Let $A(s, s^{-1}) \in \mathrm{PGL}_r(k[s, s^{-1}])$ be our transition matrix; we can simply lift it to $\mathrm{GL}_{r+1}(k[s, s^{-1}])$ to define a vector bundle E such that $P(E)$ is our original projective bundle. The reason this works for $\mathbb{P}^1$ is that there is only one transition matrix we have to worry about; for other spaces, we need to make sure that the representatives for each transition matrix still satisfy the cocycle condition. □

Remark 2.16. One thing we would like to note is that even though it seems like we can always normalize to use a representative matrix of determinant one, this is not always the case. For example,

$$\begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}$$

defines a vector bundle and hence a projective bundle, but is not equivalent in PGL_1 to a matrix of determinant one.

Remark 2.17. The matrices from (1.15) seem to provide a good example of a set of matrices in PGL_1 that do not lift to GL_2 while still verifying the cocycle condition. However, defining

$$t_{13} = -t_{24} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad t_{12} = t_{34} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \quad t_{14} = t_{23} = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

in fact verifies the cocycle condition. This is actually somewhat surprising, given that the matrices t_{12}, t_{13}, t_{23} define a projective representation of the Klein 4-group, and this representation does not lift to a linear representation [Kar87, Ex. 2.3.2]. The reason that we can still consistently lift these matrices, then, stems from the fact that unlike for the projective representation of the Klein 4-group, we do not have to consider *all* possible products of these matrices, only the ones that arise as a cocycle condition. Thus, the class of projective bundles that do not lift to vector bundles is related, but not the same, as trying to find groups G with a nontrivial Schur multiplier $H^2(G, \mathbb{C}^\times)$.

2.2. Affine bundles. For affine bundles, proving analogous statements is just a bit more involved, since while they are special cases of projective bundles, all isomorphisms of affine bundles must be isomorphisms of projective bundles that *also* fix the hyperplane $\{z_0 = 0\}$. Recall that an affine bundle is obtained as a projective bundle whose transition maps always fix a hyperplane $\{z_0 = 0\}$. So, to prove our analogue of Proposition 2.2, we lift our transition matrix in GA_{r+1} to one in GL_{r+1} by first embedding it in PGL_{r+1} and then lifting to GL_{r+1} as in (1.7); note that again, this is only possible because we are working over $\mathbb{P}^1$, on which all projective bundles lift to vector bundles as proven in the previous section. The fact that this lifted matrix must fix $\{z_0 = 0\}$, though, gives that the matrix is of the form

$$\left(\begin{array}{c|ccc} * & 0 & \cdots & 0 \\ \hline * & & & \\ \vdots & & A & \\ * & & & \end{array}\right) \tag{2.18}$$

where $A \in \mathrm{GL}_r$.

Proposition 2.19. *Every affine bundle on $\mathbb{A}^1$ is trivial.*

Proof. As in Proposition 2.2, assume our affine bundle is trivial on $D(f_1), D(f_2)$. The same argument from before lets us reduce to the case where the matrix (2.18) has the form

$$\left(\begin{array}{c|ccc} f_1^m f_2^n & 0 & \cdots & 0 \\ \hline v_1 & 1 & & 0 \\ \vdots & & \ddots & \\ v_r & 0 & & 1 \end{array}\right)$$

where $v_i \in k[s]$, for some $m, n \in \mathbb{Z}$. Then

$$\left(\begin{array}{c|ccc} f_1^{-m} & 0 & \cdots & 0 \\ \hline -v_1 f_1^{-m} f_2^{-n} & 1 & & 0 \\ \vdots & & \ddots & \\ -v_r f_1^{-m} f_2^{-n} & 0 & & 1 \end{array}\right)\left(\begin{array}{c|ccc} f_1^m f_2^n & 0 & \cdots & 0 \\ \hline v_1 & 1 & & 0 \\ \vdots & & \ddots & \\ v_r & 0 & & 1 \end{array}\right)\left(\begin{array}{c|ccc} f_2^{-n} & 0 & \cdots & 0 \\ \hline 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{array}\right) = \mathrm{Id},$$

and then we proceed by gluing step by step as before ☐

Now we would like to prove an analogue of Theorem 2.1 for affine bundles. Of course, we do not have a notion of direct sum of affine bundles, but we can still ask when our transition can be diagonalized, i.e., when an affine bundle is actually a vector bundle. This produces the following result:

Theorem 2.20. *Suppose we have an affine bundle, which we view as a projective bundle B that has transition matrix holding the hyperplane $\{z_0 = 0\}$ fixed. Let $\tilde{B}$ be its lift. Let L be the line bundle associated to the coordinate z_0. Then, B is a vector bundle if and only if $\tilde{B} \cong L \oplus \tilde{B}/L$.*

Proof. $\tilde{B}$ has transition matrix of the form

$$\left(\begin{array}{c|cccc} s^n & 0 & 0 & \cdots & 0 \\ \hline v_1 & a_{11} & a_{12} & \cdots & a_{1r} \\ v_2 & a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_r & a_{r1} & a_{r2} & \cdots & a_{rr} \end{array}\right) \tag{2.21}$$

with determinant a unit in $k[s, s^{-1}]$, by Corollary 2.15 and the discussion at the beginning of §3. Then, by applying the method of Proposition 2.6, $\tilde{B}$ is isomorphic to the vector bundle with matrix of the form

$$\left(\begin{array}{c|cccc} s^n & 0 & 0 & \cdots & 0 \\ \hline v_1 & s^{d_1} & & & \\ v_2 & & s^{d_2} & & \\ \vdots & & & \ddots & \\ v_r & & & & s^{d_r} \end{array}\right)$$

By attempting to use the inductive step in the proof of Proposition 2.6, we can moreover assume that each $v_i = \sum_{j=1-n}^{d_i-1} b_{ij} s^j$, since that inductive step keeps the hyperplane $\{z_0 = 0\}$ fixed. Finally, we see that by Theorem 2.1 and the definition of a quotient bundle, $L \cong \mathcal{O}(-n)$ and $\tilde{B} \cong \mathcal{O}(-d_1) \oplus \cdots \oplus \mathcal{O}(-d_r)$.

Now B is a vector bundle if and only if $v_1 = v_2 = \cdots = v_r = 0$ in (2.21). But by the definition of quotient bundle and direct sum, this is equivalent to $\tilde{B} \cong L \oplus \tilde{B}/L$. □

We hope that something like this criterion holds for other base spaces X, and that there is a cohomological interpretation of this statement.

Part 2

Principal bundles over schemes

CHAPTER 3

Preliminaries

In this second half of this thesis, we move to the setting of schemes, and principal bundles over them. We will see later that this is a generalization of the previous setting of varieties and fiber bundles.

For completeness, we gather here some general definitions and facts concerning group schemes and group actions on schemes. In the following, by a scheme X/S we mean a scheme X with a morphism $X \to S$ of schemes. If $S = \operatorname{Spec} R$ is affine, we say X/R. 1_X refers to the identity morphism $X \to X$, and if $f : X_1 \to X_2$, $g : Y_1 \to Y_2$ are S-morphisms of schemes, then $f \times g : X_1 \underset{S}{\times} Y_1 \to X_2 \underset{S}{\times} Y_2$ is their product, and if $X_1 = Y_1 = Z$, $(f, g) : Z \to X_2 \underset{S}{\times} Y_2$ is their pullback. If Z, X_2, Y_2, S are affine, then (f°, g°) refers to the corresponding pushout of algebra homomorphisms. Varieties will refer to reduced, separated schemes of finite type over a field k.

1. Group schemes and algebraic groups

In this section we define group schemes and algebraic groups. For more discussion, see [GIT, Ch. 0; Bor91, Ch. I].

Definition 3.1. A *group scheme* G/S is a morphism $\pi : G \to S$ with S-morphisms $\mu : G \underset{S}{\times} G \to G$ (the *group law*), $\beta : G \to G$ (the *inverse morphism*), $e : S \to G$ (the *identity morphism*) such that the following diagrams commute:

$$\begin{array}{ccc} G \underset{S}{\times} G \underset{S}{\times} G & \xrightarrow{\mu\times 1_G} & G \underset{S}{\times} G \\ {\scriptstyle 1_G\times\mu}\big\downarrow & & \big\downarrow{\scriptstyle \mu} \\ G \underset{S}{\times} G & \xrightarrow{\mu} & G \end{array}$$
(Associativity)

$$\begin{array}{ccc} G & \xrightarrow{(\beta,1_G)} & G \underset{S}{\times} G \\ {\scriptstyle (1_G,\beta)}\big\downarrow & \overset{e\circ\pi}{\searrow} & \big\downarrow{\scriptstyle \mu} \\ G \underset{S}{\times} G & \xrightarrow{\mu} & G \end{array}$$
(Inverse)

$$\begin{array}{ccc} G & \xrightarrow{(e\circ\pi,1_G)} & G \underset{S}{\times} G \\ {\scriptstyle (1_G,e\circ\pi)}\big\downarrow & \overset{1_G}{\searrow} & \big\downarrow{\scriptstyle \mu} \\ G \underset{S}{\times} G & \xrightarrow{\mu} & G \end{array}$$
(Identity)

Let H/S be another group scheme with morphisms μ', β', e'. A *morphism* of group schemes over S is an S-morphism of schemes $f : H \to G$ such that the diagrams below commute:

$$\begin{array}{ccc} H \underset{S}{\times} H & \xrightarrow{f\times f} & G \underset{S}{\times} G \\ {\scriptstyle \mu'}\big\downarrow & & \big\downarrow{\scriptstyle \mu} \\ H & \xrightarrow{f} & G \end{array} \qquad \begin{array}{ccc} H & \xrightarrow{f} & G \\ {\scriptstyle \beta'}\big\downarrow & & \big\downarrow{\scriptstyle \beta} \\ H & \xrightarrow{f} & G \end{array} \qquad \begin{array}{ccc} H & \xrightarrow{f} & G \\ {\scriptstyle e'}\nwarrow & & \nearrow{\scriptstyle e} \\ & S & \end{array} \tag{3.2}$$

An *algebraic group* G over a field k is a group scheme G/k which is a reduced, separated scheme of finite type over k, i.e., a group scheme that is also a variety over k.

Affine group schemes, that is, group schemes G/R that are affine over some ring R are what we are most interested in, especially $R = k$ is a field and G/R is in fact a variety, i.e., when it is even an algebraic group. For affine group schemes over a (commutative) ring R, the diagrams above correspond to the diagrams

$$\begin{array}{ccc} A \otimes_R A \otimes_R A & \xleftarrow{\mu^\circ \otimes 1_A} & A \otimes_R A \\ {\scriptstyle 1_A\otimes\mu^\circ}\uparrow & & \uparrow{\scriptstyle \mu^\circ} \\ A \otimes_R A & \xleftarrow{\mu^\circ} & A \end{array} \qquad \begin{array}{ccc} A & \xleftarrow{(\beta^\circ, 1_A)} & A \otimes_R A \\ {\scriptstyle (1_A,\beta^\circ)}\uparrow & \nwarrow{\scriptstyle \pi^\circ\circ e^\circ} & \uparrow{\scriptstyle \mu^\circ} \\ A \otimes_R A & \xleftarrow{\mu^\circ} & A \end{array} \qquad \begin{array}{ccc} A & \xleftarrow{(\pi^\circ\circ e^\circ, 1_A)} & A \otimes_R A \\ {\scriptstyle (1_A,\pi^\circ\circ e^\circ)}\uparrow & \nwarrow{\scriptstyle 1_A} & \uparrow{\scriptstyle \mu^\circ} \\ A \otimes_R A & \xleftarrow{\mu^\circ} & A \end{array}$$

(Associativity′) (Inverse′) (Identity′)

in the category of R-algebras, where morphisms are R-algebra homomorphisms.

Remark 3.3. R-algebras A with morphisms $\mu^\circ, \pi^\circ, e^\circ$ as above are called *Hopf algebras,* and these form a category opposite to that of affine group schemes over R with morphisms those that satisfy diagrams dual to (3.2). The study of affine group schemes is a complicated one, in that there are multiple points of view being used simultaneously: an algebraic group can be thought of as a variety with group actions or dually as a Hopf algebra. The last point of view, that of looking at the functors these schemes represents, is for example the main viewpoint of [DG70]. See [Mil12, Preface] for a discussion.

The idea is that group schemes are the scheme-theoretic version of algebraic groups, and so at least for varieties, we can think of the axioms above as defining operations on R-valued points to give us the same intuition as for Lie groups.

We redefine our primary examples from Chapter 1, §1 in this setting.

Example 3.4. Our first example is $\mathrm{GL}_r(R)$, the *general linear group of degree r* over a ring R. If R is understood, we abbreviate this as GL_r. This is

$$\pi\colon \operatorname{Spec} \frac{R[a_{11}, a_{12}, \dots, a_{rr}, t]}{\det(a_{ij})t - 1} \to \operatorname{Spec} R,$$

with the morphism induced by the inclusion $\pi^\circ\colon R \to R[a_{11}, a_{12}, \dots, a_{rr}, t]/(\det(a_{ij})t - 1)$. GL_r is then the open subset of $\mathbb{A}_R^{r^2}$ where $\det(a_{ij}) \in R^\times$. In this case, defining

$$e^\circ(a_{ij}) = \delta_{ij}, \quad \mu^\circ(a_{ij}) = \sum_h a_{ij} \otimes a_{hj}, \quad \beta^\circ(a_{ij}) = (-1)^{i+j} \det(a_{ij})^{-1} \det(a_{rs})_{r\neq j, s\neq i} \tag{3.5}$$

and extending linearly turns GL_r into a group scheme. We can think of the R-valued points in $\mathrm{GL}_r(R)$ as the set of $r \times r$ invertible matrices over R, in which case the R-algebra homomorphisms given above correspond to the usual operations on GL_r as a group.

Example 3.6. Similarly, we can define $\mathrm{PGL}_r(R)$, the *projective general linear group of degree r* over R. It is formed by taking the open subset of $\mathbb{P}^{r^2+2r}$ where $\det(a_{ij}) \neq 0$, where we note the determinant is a homogeneous polynomial in the a_{ij}. Note then that this is an affine scheme $\operatorname{Spec} R[a_{ij}]_{(\det)}$, where $R[a_{ij}]_{(\det)}$ is the zeroeth degree of the ring $R[a_{ij}][1/\det]$, and the group morphisms defined similarly to (3.5).

Definition 3.7. Let $i\colon H \to G$ be a morphism of group schemes over S. H is a *closed subgroup scheme* of G if i is a closed immersion. H is moreover a *closed algebraic subgroup* of G if both H, G are algebraic groups. By definition, a closed algebraic subset H of G is a closed algebraic subgroup if and only if it is a subgroup of G.

Example 3.8. The closed algebraic subgroups of $\mathrm{GL}_r(k)$ are called *linear algebraic groups*. Many classical groups are examples of these. For example, SL_r is the subgroup of GL_r defined by the ideal $\det(T_{ij}) = 1$.

Example 3.9. The last example of a closed algebraic subgroup we give is $\mathrm{GA}_r(R)$, the *general affine group of degree r*. It is formed by taking the closed subgroup of $\mathrm{PGL}_r(R)$ such that $T_{12} = T_{13} = \cdots = T_{1r} = 0$.

The simplest examples of groups, that is, finite groups, also have a scheme-theoretic analogue:

Example 3.10 (Constant group scheme). Let G be a finite group; it can be turned into a group scheme over a ring R that has only 0, 1 as idempotents by taking the disjoint union $G_R = \coprod_{|G|} \operatorname{Spec} R$ of $|G|$ copies of $\operatorname{Spec} R$. G_R then is equal to $\operatorname{Spec} R \times \cdots \times R = \operatorname{Spec} R[e_g]_{g\in G}$. The morphism

$$\mu^\circ\colon R \to R[e_g]_{g\in G} \otimes_R R[e_g]_{g\in G} \quad e_g \mapsto \sum_{g=ab} (e_a \otimes e_b)$$

gives a group law. Inverse is defined by $\beta^\circ\colon e_g \mapsto e_{g^{-1}}$ and identity by

$$e\colon R[e_g]_{g\in G} \to R \quad e_g \mapsto \delta_{g1},$$

i.e., $e_g \mapsto 1$ if $g = 1 \in G$, 0 otherwise. See [Tat97, 2.10] for details.

2. Group actions on schemes

We can now talk about group actions on schemes, keeping in mind that we want to generalize our notion of fiber bundles from before.

Definition 3.11. A group scheme G/S *acts (on the left)* of a scheme X/S if an S-morphism $m_X\colon G \underset{S}{\times} X \to X$ is given, such that

(a) The diagram below commutes, where μ is the group law for G:

$$\begin{array}{ccc} G \underset{S}{\times} G \underset{S}{\times} X & \xrightarrow{1_G \times m_X} & G \underset{S}{\times} X \\ \downarrow{\scriptstyle \mu\times 1_G} & & \downarrow{\scriptstyle m_X} \\ G \underset{S}{\times} X & \xrightarrow{m_X} & X \end{array}$$

(b) The composition below is the identity 1_X, where e is the identity morphism for G:

$$X \xrightarrow{\sim} S \underset{S}{\times} X \xrightarrow{e\times 1_X} G \underset{S}{\times} X \xrightarrow{m_X} X$$

Similarly, we can define the notion of G/S that *acts on the right* of X/S. If the action is understood, we call X a G-scheme.

Let Y/S be another G-scheme with action $m_Y\colon G\underset{S}{\times}Y \to Y$, and let $f\colon X\to Y$ be a morphism over S. f is then a *morphism of G-schemes* or a *G-equivariant morphism* if the diagram below commutes:

$$\begin{array}{ccc} G\underset{S}{\times}X & \xrightarrow{(1_G,f)} & G\underset{S}{\times}Y \\ {\scriptstyle m_X}\downarrow & & \downarrow{\scriptstyle m_Y} \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

We return to our favorite examples:

Example 3.12 ($\mathrm{GL}_r(R)$ acting on $\mathbb{A}^r_R$). We want to define an action $\mathrm{GL}_r(R)\underset{R}{\times}\mathbb{A}^r_R \to \mathbb{A}^r_R$. Let $\mathbb{A}^r_R = R[x_1,\dots,x_n]$; we can define this action as the morphism on schemes corresponding to the R-algebra homomorphism

$$R[x_1,\dots,x_n] \to \frac{R[a_{ij}][t]}{\det(a_{ij})t-1}\underset{R}{\otimes}R[x_1,\dots,x_n]$$
$$x_i \mapsto \sum_j a_{ij}\otimes x_j$$

Example 3.13 ($\mathrm{PGL}_r(R)$ acting on $\mathbb{P}^r_R$). We want to define an action $\sigma\colon \mathrm{PGL}_r(R)\underset{R}{\times}\mathbb{P}^r_R \to \mathbb{P}^r_R$. This is a bit trickier, but recall from [Har77, II, Thm. 7.1] that the morphism σ can be given by finding an invertible sheaf on $\mathrm{PGL}_r(R)\underset{R}{\times}\mathbb{P}^r_R$ that is generated by global sections $s_0,\dots,s_r$, and then by the rule $\sigma^*(x_i)=s_i$. By [EGAII, §4.2], $p_1^*[\mathcal{O}_{\mathbb{P}^{r^2+2r}}(1)]\otimes p_2^*[\mathcal{O}_{\mathbb{P}^r}(1)]$ is such an invertible sheaf, and we can define σ via

$$\sigma^*(\mathcal{O}_{\mathbb{P}^r}(1)) \cong p_1^*[\mathcal{O}_{\mathbb{P}^{r^2+2r}}(1)]\otimes p_2^*[\mathcal{O}_{\mathbb{P}^r}(1)]$$
$$\sigma^*(x_i) = \sum_{j=0}^{r} p_1^*(a_{ij})\otimes p_2^*(x_j).$$

The idea of group actions on schemes suggests that we might be able to take their quotients by such an action. It is easy to write down a definition:

Definition 3.14. The *quotient scheme of X by G* is a scheme Y and a morphism $p\colon X\to Y$ of schemes such that

(1) p is invariant under G, i.e., $p\circ m_X(g,-)=p$ for all $g\in G$, and
(2) (Y,p) have the following universal property: for every scheme Z/S, and every G-invariant morphism $f\colon X\to Z$, there is a unique morphism $h\colon Y\to Z$ that makes the diagram

$$\begin{array}{ccc} X & & \\ {\scriptstyle p}\downarrow & \searrow^{f} & \\ Y & \overset{h}{\dashrightarrow} & Z \end{array}$$

commute.

However, there is no reason to expect that such an object exists; indeed, this is the main topic of [GIT]. On the other hand, if G_R is a constant group scheme associated to a finite group G as in Example 3.10, we have the following nice criterion due to Chevalley:

THEOREM 3.15 ([SGA1, Exp. V, Prop. 1.8]). *Let X be a scheme on which a constant group scheme G acts on the right. The quotient of X by G exists if and only X is the union of affine open sets that are invariant under the action of G, or in other words, every orbit of G in X is contained in an affine open set.*

On a G-invariant affine open subset of X, since G acts on the right of X, we see that G acts on the left of the coordinate ring A. Thus, the statement above really boils down to

PROPOSITION 3.16 ([Bou64, Ch. V, §1, n° 9, Prop. 22; §2, n° 2, Thm. 2]). *Let A be a ring on which a finite group G acts on the left, $B = A^G$ the subring of invariants of A, $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, $p\colon X \to Y$ the morphism associated to the inclusion, invariant under G. Then,*

(1) A is integral over B, i.e., p is an integral morphism;
(2) p is surjective, its fibers are the orbits of G, and the topology on Y is the quotient topology induced by X;
(3) Letting $x \in X$, $y = p(x)$, G_x the stabilizer of x, then $k(x)$ is a normal ("quasi-Galois") extension of $k(y)$ and the morphism $G \to \operatorname{Gal}(k(x)/k(y))$ is surjective;
(4) (Y, p) is the quotient scheme of X by G.

The idea is then to take these affine schemes and glue them together using the uniqueness of the pair (Y, p); see [SGA1, Exp. V] for details.

Of course there are examples where a finite group acts on a scheme, but there does not exist a quotient scheme. The canonical example where this occurs is the following:

EXAMPLE 3.17 ([Hir62]). Let $V_0 = \mathbb{P}^r_{\mathbb{C}}$ be complex projective three-space, and γ_1 and γ_2 two conics that intersect normally in exactly two points P_1 and P_2. For $i = 1, 2$, construct $\overline{V}_i$ by first blowing up γ_i, and then γ_{3-i} in the result. Let V_i be the open set in $\overline{V}_i$ lying over $(V_0 - P_{3-i})$; by gluing together $\overline{V}_i$ along along V_i, we get a scheme U. Letting $\sigma_0\colon V_0 \to V_0$ be the projective transformation interchanging P_1 and P_2, γ_1 and γ_2, σ_0 induces an automorphism $\sigma\colon U \to U$ of order 2. Hironaka in [Hir62] showed that this scheme under the action $G = \{1, \sigma\}$ has no quotient in the sense above, but it does have the structure of what is called an *algebraic space* [Knu71, p. 15–16].

CHAPTER 4

Principal bundles in étale topology

Our proof of Theorem 2.1 relied on the fact that holomorphic vectors bundles on the variety $\mathbb{P}^1$ are isomorphic to algebraic ones. But this is not always the case for fiber bundles with different structure groups, as we saw in Example 1.14. But having to talk about holomorphic bundles separately from algebraic ones is inconvenient given that there is then no unified way to talk about holomorphic bundles and algebraic bundles on algebraic varieties defined over $\mathbb{C}$, and also we would like to have an analogue of holomorphic bundles for fields that are not $\mathbb{C}$. The solution to this is to use what is called the étale topology; even though this is not strictly a topology, it provides a framework in which to talk about holomorphic bundles algebraically. We discuss first what is the étale topology, and use it to describe that for structure group G being GL_r or GA_r, fiber systems are locally trivial in étale topology if and only if they are in locally trivial in Zariski topology, but not for $G = \mathrm{PGL}_r$.

1. Differential forms; unramified and étale morphisms

The idea of an étale morphism is that we want an analogue of a local isomorphism of complex analytic manifolds, which in particular induces an bijection of tangent spaces, and so it becomes important to talk about tangent bundles. In the algebraic world, though, it is more convenient to talk about its dual, the sheaf of differentials. So we first review the algebraic theory of differentials here, following [Har77, II.8].

Let A be a ring, B an A-algebra, and M an B-module. An *A-derivation D* of B into M is an additive map $B \to M$ such that $d(bb') = bd(b') + bd(b')$, and $d(a) = 0$ for all $a \in A$.

Definition 4.1. The *module of relative differential forms* of B over A to be a B-module $\Omega_{B/A}$, together with an A-derivation $d_{B/A} : B \to \Omega_{B/A}$, which satisfies the following universal property: for any B-module M, and for any A-derivation $d' : B \to M$, there exists a unique B-module homomorphism $f : \Omega_{B/A} \to M$ such that $d' = f \circ d_{B/A}$.

To construct such a module, consider the "diagonal" morphism $B \otimes_A B \to B$ defined by $b \otimes b' \mapsto bb'$, and let I be the kernel of this morphism. Consider $B \otimes_A B$ as a B-module by multiplication on the left. Then I/I^2 has the structure of a $B \otimes_A B$-module; denote $\Omega_{B/A}$ to be the B-module obtained by restriction of scalars. Define the map $d_{B/A} : B \to I/I^2$ by $d_{B/A}(b) = 1 \otimes b - b \otimes 1$ (modulo I^2).

We have the following facts about $\Omega_{B/A}$:

Proposition 4.2 ([Mat70, p. 186]). *If A' and B are A-algebras, and $B' = B \otimes_A A'$, then $\Omega_{B'/A'} \cong \Omega_{B/A} \otimes_B B'$.*

Proposition 4.3 ([Mat70, Thm. 57]). *Let $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ be rings and homomorphisms. Then there is a natural exact sequence of C-modules*

$$\Omega_{B/A} \otimes_B C \xrightarrow{v} \Omega_{C/A} \xrightarrow{u} \Omega_{C/B} \to 0.$$

We can then globalize this definition to define sheaves over our space. Let X be a scheme over Y, and $\Delta_{X/Y} = \Delta$ the diagonal morphism $X \to X \times_Y X$. This is an immersion (i.e., the composition of a closed then open immersion), and so $\Delta(X)$ is a closed subscheme of V an open subscheme of $X \times_Y X$.

Definition 4.4. Let $\mathscr{J}_X$ be the ideal sheaf corresponding to the closed subscheme $\Delta(X) \subset V$. $\Delta^*(\mathscr{J}_X/\mathscr{J}_X^2)$ is then a quasi-coherent sheaf on X, which we denote $\Omega_{X/Y}$; this is the *sheaf of relative differential forms of X over Y*.

If $Y = \operatorname{Spec} A$ and $X = \operatorname{Spec} B$, then $\Omega_{X/Y} \cong \widetilde{\Omega_{B/A}}$, and in the general case, these glue together to give $\Omega_{X/Y}$. This means that each of the commutative algebraic results above have sheaf-theoretic analogues which we use below.

In the following, we assume that all our schemes are locally noetherian. The definitions here are from [SGA1, I].

Definition 4.5. Let $f : X \to Y$ be a morphism locally of finite type, and let $x \in X$, $y = f(x)$. f is *unramified* if $(\Omega_{X/Y})_x = 0$ for all $x \in X$.

f is *étale* if it is flat, i.e., $\mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is flat for all $x \in X$, and unramified.

A finite family of étale morphisms $\{f_i : U_i \to Y\}$ where U_i are affine is an *étale cover* if the image of the f_i cover Y.

Unramified and étale morphisms work nicely under composition and base change:

Proposition 4.6. *The composition of unramified (resp. étale) morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ is unramified (resp. étale); the base change of an unramified (resp. étale) morphism is unramified (resp. étale).*

Proof. This follows by Propositions 4.2 and 4.3 for unramified maps; for flatness, this is [Mat70, 3.B and 3.C]. □

We also note that we have an alternate description of unramified morphisms:

Theorem 4.7 ([EGAIV$_4$, Thm. 17.4.1]). *Let $f : X \to Y$ be a morphism locally of finite type. Then, f is unramified if and only if it is quasi-finite and the ring $\mathscr{O}_{X,x}/\mathfrak{m}_y\mathscr{O}_{X,x}$ is a field for all $x \in X$, $y = f(x)$, and is a finite separable extension of the residue field $k(y)$.*

Definition 4.8. We call the degree of the field extension $k(y) \to \mathscr{O}_{X,x}/\mathfrak{m}_y\mathscr{O}_{X,x}$ the *degree* of our unramified morphism. It is locally constant, hence constant on a connected component of Y.

Now an étale cover is *galois* if it is of the form $X \to X/\mathfrak{g}$ for a finite group $\mathfrak{g}$ of automorphisms of X. This happens if and only if $\mathfrak{g}$ acts freely on X [Ser58a, n° 1.4]. Note that by Theorem 4.7, we can think of this as the scheme-theoretic version of a galois extension of fields. Now let $\{f_i : U_i \to Y\}$ be an étale cover; we claim there is an associated galois cover:

Proposition 4.9 ([Ser58a, n° 1.4]). *Let Y be connected, and $f : X \to Y$ a surjective étale morphism, i.e., an étale cover of Y, of degree n. Then, there exists a scheme $\pi : Z \to Y$ such that f is an galois cover, and factors through f.*

Proof Sketch. Let $f : X \to Y$ be étale; then $f^{\times n} : X^{\times n} \to Y^{\times n}$ is a surjective étale cover of degree n^n by Proposition 4.6. There is then a cartesian diagram

$$\begin{array}{ccc} X_Y^{\times n} & \xrightarrow{\tilde{\Delta}} & X^{\times n} \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f^{\times n}} \\ Y & \xrightarrow{\Delta} & Y^{\times n} \end{array}$$

$\tilde{f}$ is then a degree n^n étale cover of Y, and S^n acts on $X_Y^{\times n}$. Let Z be the inverse image through $\tilde{\Delta}$ of the open subset of $X^{\times n}$ consisting of n-tuples $(y_1, \ldots, y_n)$ where each y_i is distinct. $Z \to Y$ is then an étale cover of degree $n!$, and S_n acts freely on Z; see [Ser58a, n° 1.4] for details. The factorization comes from [Ser58a, n° 1.3(f)]. □

2. Principal and fiber bundles in étale topology and a comparison

Étale morphisms allow us to talk about a new "topology," namely, instead of thinking of open sets in X, we allow "open covers" which are actually just étale covers. This generalizes the notion of an open covering, and gives us a new class of fiber bundles which are not necessarily algebraic, i.e., locally trivial in Zariski topology.

Recalling our definitions from Chapter 1, we have the following analogue of fiber bundles in this new topology:

Definition 4.10. A fiber system $E \to X$ is *locally trivial in étale topology* if there exists an étale cover $\{f_i : U_i' \to U_i\}$ of X such that $f_i^* E|_{U_i}$ are isomorphic to the trivial bundle.

We recall Example 1.14, which gives an example of a PGL-bundle that is locally trivial in Zariski topology but not in the étale topology.

Example 4.11 (See Example 1.14). Let

$$A = \mathbb{C}[s, s^{-1}, t, t^{-1}], \quad B = \mathbb{C}[s^{1/2}, s^{-1/2}, t^{1/2}, t^{-1/2}].$$

Then $A \hookrightarrow B$ as rings. This is a flat ring homomorphism, for B is the free module over A generated by $1, s^{1/2}, t^{1/2}, s^{1/2}t^{1/2}$. We have $\ker B \otimes_A B = I = 0$, hence $\Omega_{B/A} = 0$. Thus, the map $\operatorname{Spec} B \to \operatorname{Spec} A$ on spectra is an étale cover of $\operatorname{Spec} A$, and the map defined in (1.14) is then a local trivialization of the bundle in this étale cover. Hence Example 1.14 is locally trivial in étale topology.

In contrast, however, vector bundles that are locally trivial in étale topology are actually locally trivial in étale topology. To prove this, we change our point of view slightly to principal bundles.

Definition 4.12. Let $P \to X$ be a fiber system with structure group G as we had before. P is a *principal G-bundle locally trivial in Zariski (resp. étale topology)* if there is an open cover U_i of X (resp. an étale cover $\{f_i : U_i \to X\}$ of X) such that $P|_{U_i} \cong U_i \times G$ (resp. $f_i^* P \cong U_i \times G$).

Note this is a special case of a fiber system such that our fibers F are isomorphic to G itself. If F is a fixed G-scheme, then principal G-bundles and fiber bundles with structure group G and fiber F are in one-to-one correspondence, since we can just take our transition functions and use them to patch together a principal G-bundle by having them act on G. We remind the reader of our discussion in Remark 1.13: these objects are nicer to deal with because there are canonical ways to choose morphisms $X \to G$ that represent automorphisms of the trivial bundle.

Moreover, we have the nice characterization for when the following criterion for principal bundles makes them even better because there is a principal bundles are locally trivial:

Proposition 4.13. *Let G be connected. A principal G-bundle $\pi : P \to X$ is locally trivial if and only if there exists a Zariski local (resp. étale local) section at each $x \in X$. Note an étale local section is an étale morphism $f : U \to X$ with x in its image and a morphism $\sigma : U \to U \times P$ such that $f \circ \sigma' = \pi$.*

Proof. If U is an open set (resp. $f : U \to X$ is an étale morphism) on which π is trivial, then clearly we are done. In the other direction, if σ is a Zariski local section on U (resp. σ is an étale local section) then $\Psi(x, g) = \sigma(x) \cdot g$ defines an isomorphism $U \times G \to P$ (resp. $U \times G \to f^*UP$), where the inverse is as defined in [Ser58a, Prop. 2]. Note in particular that this morphism is bijective by the fact that G acts effectively on itself since it's connected. □

Remark 4.14. We note that this is a feature specific to principal bundles that does not apply to fiber bundles. For example, every vector bundle has the section defined by choosing the origin in each fiber.

Proposition 4.15 ([Ser58a, Prop. 1]). *The isomorphism classes of principal G-bundles over X that are trivial after base change through a Galois cover $X' \to X$ with associated group $\mathfrak{g}$ of automorphisms is in bijection with the (non-abelian) group cohomology $H^1(\mathfrak{g}, \Gamma(X', G))$, where $\Gamma(X', G)$ is the group of morphisms $X' \to G$, on which $\mathfrak{g}$ acts by $(\sigma f)(x') = f(x' \cdot \sigma)$.*

Proof. We have the commutative diagram

$$\begin{array}{ccc} X' \times G & \longrightarrow & P \\ \big\downarrow{\scriptstyle \pi'} & & \big\downarrow \\ X' & \longrightarrow & X \end{array}$$

P is identified with $P' = X' \times P$ quotiented out by $\mathfrak{g}$ by the above. The action of $\mathfrak{g}$ must be compatible with the projection π', and so we have

$$(x', g) \cdot \sigma = (x' \cdot \sigma, \varphi_\sigma(x') \cdot g), \quad \sigma \in \mathfrak{g}$$

for some $\varphi_\sigma : X' \to G$, and so φ_σ is a 1-cochain of $\mathfrak{g} \to \Gamma(X', G)$. The associativity of the action of $\mathfrak{g}$

$$(x', g) \cdot \sigma\tau = ((x', g) \cdot \sigma) \cdot \tau, \quad \sigma, \tau \in \mathfrak{g}$$

implies the identity

$$\varphi_{\sigma\tau}(x') = \varphi_\tau(x' \cdot \sigma) \cdot \varphi_\sigma(x'),$$

which is exactly the cocycle condition. Conversely, such a cocycle gives a well defined operation of $\mathfrak{g}$ on $X \times G$, hence defines P. Finally, this implies that two cocycles correspond to the same bundle if and only if they are cohomological. □

We can now prove

Theorem 4.16 ([Ser58a, Thm. 2], [Ser58b, nº 15]). *A principal bundle $P \to X$ with group GL_r is locally trivial in étale topology if and only if it is locally trivial in Zariski topology.*

Proof. By restriction, we can assume that P is étale-trivial on X, and so let $\pi : U \to X$ be a galois cover with group $\mathfrak{g}$, such that π^*P is trivial. Let A_x be the semilocal ring for $\pi^{-1}(x)$ in U. The group $\Gamma_x(U, \mathrm{GL}_r)$ of germs of morphisms $\pi^{-1} \to \mathrm{GL}_r$ can be identified with $\mathrm{GL}_r(A_x)$. By the Proposition above, P defines an element $p_x \in H^1(\mathfrak{g}, \mathrm{GL}_r(A_x))$; it suffices to show that this is trivial, i.e., $H^1(\mathfrak{g}, \mathrm{GL}_r(A_x)) = 0$.

Now for $x \in X$, let $y_1 \in \pi^{-1}(x)$. Choose h an $n \times n$ matrix in A_x that is the identity on y_1 but zero on other points of $\pi^{-1}(x)$. If $\varphi_\sigma \in H^1(\mathfrak{g}, \mathrm{GL}_r(A_x))$, then put

$$a = \sum_{\tau \in \mathfrak{g}} \tau(h) \cdot \varphi_\tau.$$

a is then invertible on every point in $\pi^{-1}(x)$. Moreover,

$$\sigma(a)\varphi_\sigma = \sum_{\tau \in \mathfrak{g}} \sigma\tau(h) \cdot \sigma(\varphi_\tau)\varphi_\sigma = \sum_{\tau \in \mathfrak{g}} \sigma\tau(h) \cdot \varphi_{\sigma\tau} = a,$$

and so φ_σ is a coboundary element as desired. □

The proof above constructs a section $X \to P$ of our principal bundle, by proving that the section $U \to \pi^*P$ defined by a is in fact invariant under the action of $\mathfrak{g}$, hence descends to a morphism $X \to P$ by the property of quotient schemes. We can do the same thing with an affine bundle; indeed, the same construction above gives a section by using $\varphi_\sigma \in H^1(\mathfrak{g}, \mathrm{GA}_r(A_x))$ instead.

Remark 4.17. Groups that satisfy the property in Theorem 4.16 are called *special*. For a full classification of groups of this type, see [Gro58b]. Note that specialness is preserved under group extensions [Ser58a, Lem. 6; Gro58a, Prop. 5.3.1], giving another way to prove that GA_r is special.

We recall that the analogue for Theorem 4.16 does not hold for projective bundles. Tsen's theorem 1.16 from before can be used to prove that for projective bundles over curves, though, an analogue does hold. For projective spaces, this is also true, but the proof gets much more involved. This is the subject of étale cohomology and Brauer groups.

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