F-purity and Frobenius splitting do not coincide in rigid analytic geometry

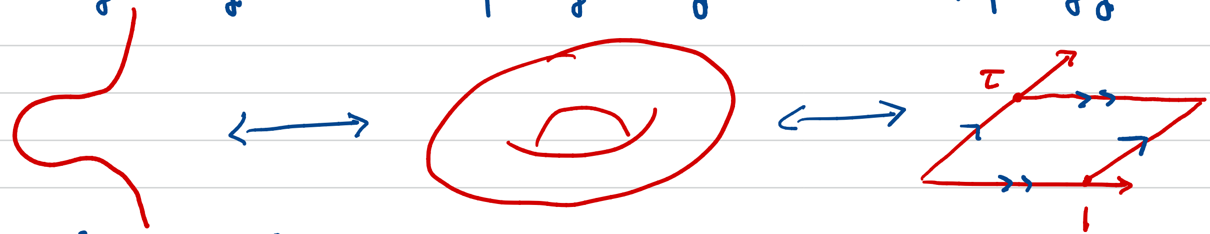
joint with Rankeya Datta

eCARs, 6/27–28
Early Commutative Algebra Researchers
https://princeton.edu/~takumim/eCARs

§1. Motivation: Uniformization of elliptic curves

Elliptic curves/$\mathbb{C}$

Algebraic geometry ⟷ Complex geometry ⟷ Topology



$\{y^2z = x^3 + axz^2 + bz^3\}$

(see Abikoff, "The uniformization theorem" 1981)

Thm [Klein 1882; Poincaré, Koebe 1907] "Uniformization"

E elliptic curve/$\mathbb{C}$

$\Rightarrow$ $\exists$ lattice $\Lambda = \mathbb{Z} + \mathbb{Z}\cdot\tau \subseteq \mathbb{C}$ for some $\tau \in \mathbb{H}$ (upper half plane) s.t.

$E \simeq \mathbb{C}/\Lambda$ (biholomorphic)

Q What happens over p-adic fields? (e.g. $\mathbb{Q}_p$ or its extensions)

Def k field. $|\cdot|: k \to \mathbb{R}_{\geq 0}$ is a valuation if:

- $|x| = 0 \iff x = 0$ "detects zero"
- $|xy| = |x|\cdot|y|$ "multiplicative"
- $|x+y| \leq |x| + |y|$ "triangle inequality"

$|\cdot|$ is non-Archimedean (NA) if

- $|x+y| \leq \max\{|x|, |y|\}$ "ultrametric inequality"

Ex $\mathbb{Q}$

- $|\cdot|_\infty$ usual "Archimedean" valuation $\xrightarrow{\text{complete}} \mathbb{R} \xrightarrow{\text{alg. cl.}} \mathbb{C}$
- $|\cdot|_p$ p-adic valuation for primes $p > 0$ $\xrightarrow{\text{complete}} \mathbb{Q}_p \xrightarrow{\text{alg. cl.}} \mathbb{C}_p$ "p-adic numbers"

$$\left|p^n \tfrac{a}{b}\right| = p^{-n} \quad (p \nmid a,\ p \nmid b)$$

"Formal Laurent series"

$$\mathbb{F}_p((t)) = \left\{ \sum_{i \in \mathbb{Z}} a_i t^i \ :\ \begin{array}{l} a_i \in \mathbb{F}_p \\ a_i = 0\ \forall\, i \ll 0 \end{array} \right\}, \quad \left|\sum a_i t^i\right| = \max_{a_i \neq 0} \{e^{-i}\}$$

Def $(k, |\cdot|)$ is complete if k is complete as a metric space wrt $|x-y|$

Q Is every elliptic curve over $\mathbb{C}_p$ isomorphic to $\mathbb{C}_p/\Lambda$?
discrete subgroup

Can't work! $\Lambda \subseteq \mathbb{C}_p$ additive subgroup, $\lambda \in \Lambda$

$$|p^n\lambda| = p^{-n}|\lambda| \to 0 \text{ as } n \to \infty \implies \Lambda \text{ not discrete}$$

Fix

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{e^{2\pi i(\ )}} & \mathbb{C}^* \\ \downarrow & & \downarrow \\ \mathbb{C}/\Lambda & \longrightarrow & \mathbb{C}^*/q^{\mathbb{Z}} \end{array}$$

$q = e^{2\pi i \tau}$

Note $q^{\mathbb{Z}} \subseteq \mathbb{C}_p^*$ is a discrete subgroup if $|q|<1$.

Tate's uniformization theorem [1959 – 1962] (published 1995) works for any complete NA field

E elliptic curve$/\mathbb{C}_p$ w/ $|j\text{-invariant}|>1$

$\implies \exists\, q \in \mathbb{C}_p$ w/ $|q|<1$ s.t. $E(\mathbb{C}_p) \simeq \mathbb{C}_p^*/q^{\mathbb{Z}}$

What does this mean?

§2. Rigid analytic spaces [Tate 1962]

distributed by IHES with(out) Tate's permission
published 1969 (in Russian), 1971 (in English)

The unit polydisc $(k, |\cdot|)$ complete NA field "building blocks"

Def The Tate algebra of dimension n is

regular UFD, Jacobson [Tate 1962]
excellent [Kiehl 1969]

$$T_n(k) := k\{X_1, X_2, \dots, X_n\}$$

$$:= \left\{ \sum_{\nu \in \mathbb{Z}^n_{\geq 0}} a_\nu X^\nu \; : \; \begin{array}{c} a_\nu \in k \\ \lim\limits_{|\nu|\to\infty} |a_\nu| = 0 \end{array} \right\} \subseteq k[\![X_1, X_2, \dots, X_n]\!]$$

"restricted" or "strictly convergent" power series

Fact $(k = \bar{k})$ $f \in k[\![X_1, X_2, \dots, X_n]\!]$ lies in $T_n(k)$
$\iff f(x)$ converges $\forall\, x \in k^n$ s.t. $|x| \leq 1$

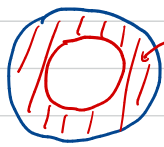
Obs $|a_\nu| \to 0 \iff \sum a_\nu < \infty$

$$\Big| \sum_{\nu_0 \leq \nu \leq \nu_1} a_\nu \Big| \leq \max_{\nu_0 \leq \nu \leq \nu_1} \{|a_\nu|\} \to 0$$

Pf of Fact $\Leftarrow$: plug in $(1, 1, \dots, 1) \in k^n$
$\Rightarrow$: $|a_\nu x^\nu| \leq |a_\nu| \to 0$ $\square$

Def The unit polydisc of dimension n is
$$\mathbb{B}^n_k := \mathrm{Max}(T_n(k)) := \{\text{maximal ideals in } T_n(k)\} \quad \text{as a set}$$
together with a sheaf $\mathcal{O}$ of functions with respect to the "G-topology".

Ex (Annulus) $k = \bar{k}$, choose $c \in k$ s.t. $0 < |c| \leq 1$

$\mathbb{B}^1_k$  $\{|c| \leq |x| \leq 1\} = \mathcal{U}$ $\qquad \mathcal{O}(\mathcal{U}) = \dfrac{k\{x, y\}}{(xy - c)}$

$\mathcal{U}$ is an example of a Laurent domain.

Rigid analytic spaces are obtained by gluing together $(\mathrm{Max}(A), \mathcal{O}_A)$'s, $T_n(k) \twoheadrightarrow A$

Applications

AG: p-adic uniformization for curves and abelian varieties
[Tate, Mumford, Raynaud, ... 1960s–1970s]

Abhyankar's conjecture on $\pi_1^{\text{ét}}$ for curves in char. $p > 0$
[Raynaud, Harbater 1994]

NT: local Langlands for GL_n — Berkovich spaces — [Harris–Taylor 2001]

weight-monodromy conjecture for complete intersections [Scholze 2012]

perfectoid spaces

CA: Hochster's homological conjectures [André 2018, ...]

Today New applications to theory of F-singularities in prime char. CA
↳ ∃ applications of Berkovich spaces [Canton 2018]

§3. F-singularities R noetherian of prime char. $p > 0$

Recall Frobenius map $F : R \longrightarrow F_* R$ (same ring w/ R-alg. structure given by F) is a ring homomorphism.
$$r \longmapsto r^p$$

Idea Use F to detect singularities of R.

Obs R reduced $\iff$ F is injective

Pf $r^m = 0 \ \exists m \geq 1 \iff r^{p^e} = 0 \ \exists e \geq 0 \iff F(r^{p^{e-1}}) = 0 \ \exists e \geq 0$ $\square$

Thm [Kunz 1969] R regular $\iff F$ faithfully flat

Pf of $\Rightarrow$ STS $\hat{R}_m \xrightarrow{F} F_* \hat{R}_m$ flat.

$$\begin{array}{ccc} \hat{R}_m & \xrightarrow{F} & F_*\hat{R}_m \\ \downarrow \wr & \text{Cohen} & \downarrow \wr \\ k[[t_1,\dots,t_d]] & \hookrightarrow & k^{1/p}[[t_1^{1/p},\dots,t_d^{1/p}]] \end{array}$$

$k[[t_1,\dots,t_d]] \hookrightarrow k^{1/p}[[t_1^{1/p},\dots,t_d^{1/p}]]$ is a free extension. $\square$

Def [Hochster–Roberts 1976] R is F-pure if F is pure, i.e. (also "universally injective")

$\forall$ R-mod's M, $id_M \otimes_R F : M \otimes_R R \longrightarrow M \otimes_R F_* R$ is injective.

[Mehta–Ramanathan 1985] R is Frobenius split if F splits as a map of R-mod's

($\exists\ \Phi : F_* R \to R$ s.t. $\Phi \circ F = id_R$)

Fact

$$\begin{array}{ccccc} \text{regular} & & & & \text{Frobenius split} \\ & \underset{\Rightarrow\text{ pure}}{\overset{\text{f.flat}}{\searrow}} & & \underset{\Rightarrow\text{ pure}}{\overset{\text{split}}{\swarrow}} & \\ & & \text{F-pure} & & \end{array}$$

Why (1) [Hara–Watanabe 2002] F-purity is the positive characteristic analogue of log canonical singularities

(2) [Hochster–Roberts 1974] Rings of invariants of reductive groups acting on regular rings are Cohen–Macaulay

$/k=\bar{k}$
(3) [Mehta–Ramanathan 1985] Vanishing theorems for Schubert varieties

(4) [Greco–Traverso 1980; Roberts–Zaare-Nahandi 1984; Datta–M] Every projective variety is birational to a weakly normal hypersurface

Q [Hochster 1970s; Smith–Zhang 2015]
Is every excellent F-pure ring Frobenius split?

Excellence: Answer is no for some non-excellent DVRs [Datta–Smith 2016]

Known Yes if F is (module-)finite [Hochster–Roberts 1976]

AG: rings eft / fields [Datta–M]

CA: complete local rings (Auslander [Fedder 1983])
rings eft / clr's [Datta–M]

versions of Hahn-Banach hold

RAG: $T_n(k)$ when $(k, |\cdot|)$ is spherically complete or $k^{1/p}$ of countable type
[Datta–M] w/ Canton and Stevenson

regular $\Rightarrow$ F-pure

Q [Hochster 1970s; Smith–Zhang 2015] also Huneke, Katzmann, Schwede

Is every excellent F-pure ring Frobenius split?

Why In tight closure theory, need test elements ⇝ test ideals, the pos. char. analogue of multiplier ideals

Proofs reduce to the case when $\underbrace{\exists \text{ nonzero maps } F_* R \to R}_{\Downarrow}$.

All known cases assume excellence. exellence for Dedekind domains [Datta–M]

Positive answer would have possibly been helpful to construct test ideals for arbitrary excellent rings.

Main Thm [Datta–M] No!

§4 **Main Result** Using some ideas of Gabber, we showed:

Thm [Datta–M] $\forall$ prime $p>0$, $\exists$ complete NA field $(k, |\cdot|)$ of char. $p>0$ s.t. $T_n(k)$ is not Frobenius split $\forall\, n \in \mathbb{Z}_{>0}$.

[Kiehl 1969] showed $T_n(k)$ is excellent, so this resolves Hochster's question.

Main idea

① Characterize Frobenius splitting in terms of $k \subseteq k^{1/p}$.

Thm [Datta–M] $T_n(k)$ Frobenius split $\forall\, n \in \mathbb{Z}_{>0}$
$\iff \exists$ nonzero continuous k-linear $\phi : k^{1/p} \to k$

Pf $\Leftarrow$: Replace ϕ by $(\phi(1)^{-1} \cdot -) \circ \phi$ to assume $\phi(1) = 1$.

$$\Phi : F_* T_n(k) \longrightarrow T_n(k) \qquad \text{is a splitting}$$

$$\sum_{\nu \in \mathbb{Z}^n_{\geq 0}} a_\nu X^\nu \longmapsto \sum_{\nu \in p\mathbb{Z}^n_{\geq 0}} \phi(a_\nu) X^{\nu/p}$$

$|\phi(a_\nu)| \to 0$ by continuity since $|a_\nu| \to 0$

$\Rightarrow$: Consider the k-linear map

$$f : F_* k \hookrightarrow F_* k\{X\} \xrightarrow[\text{splitting}]{\Phi} k\{X\} \twoheadrightarrow \frac{k\{X\}}{(X)} \xrightarrow{\sim} k$$

which is nonzero: $f(1) = 1$.

STS f is continuous. If not, $\exists\, a_i \in F_* k$ s.t. $|a_i| \to 0$ but

$$\Phi(a_i) = \sum_{j=0}^{\infty} b_{i,j} X^j, \quad |f(a_i)| = |b_{i,0}| \to \infty.$$

Choose a sequence $\{m_i\}$ s.t.

$$\max_{0 \le r \le i-1} \{ |b_{m_r, i-r}| \} < |b_{m_i,0}| \quad \text{can do this by}$$

$$\Rightarrow \left| \text{coeff. of } X^i \text{ in } \Phi\left(\sum_{i=0}^{\infty} a_{m_i} X^{ip}\right)\right| = |b_{m_i,0}| \to \infty \; \lightning \quad \square$$

② Find a field k s.t. $\nexists$ nonzero continuous $k^{1/p} \to k$

ⓐ [Gabber; Blaszczok–Kuhlmann 2015]
$\exists$ complete NA field k of char. $p>0$ and a sequence
$k \subsetneq k^{sp} \subseteq k^{1/p}$ of complete NA fields

where k^{sp} is a <u>spherical completion</u> of k
↳ $\nexists\, k^{sp} \subsetneq L$ w/ same value group + residue field as k

ⓑ [van der Put – van Tiel 1967] If $k \subsetneq k^{sp}$, then $\nexists$ continuous k-linear map $k^{sp} \to k$ } Idea to apply due to Gabber

<u>Construction for ②ⓐ [Gabber]</u>

① Choose $r_i \in \mathbb{R}_{>0}$, $r_i \searrow 0$, $\{r_i\}$ $\mathbb{Q}$-linearly independent
Set $\Gamma = \mathbb{Z} \cdot \{r_i\} \subseteq \mathbb{R}$

② $\mathbb{F}_p[t^\Gamma] := \left\{ \sum_{r \in \Gamma} a_r t^r : \{\gamma \in \Gamma : a_\gamma \neq 0\} \text{ finite} \right\}$ "generalized polynomials"

$\cap|$

$\mathbb{F}_p(t^\Gamma) := \mathrm{Frac}(\mathbb{F}_p[t^\Gamma])$ "generalized rational functions" $\quad \left|\sum a_\gamma t^\gamma\right| = \max_{a_\gamma \neq 0} \{e^{-\gamma}\}$

$\cap|$

$\mathbb{F}_p((t^\Gamma)) := \left\{ \sum_{\gamma \in \Gamma} a_\gamma t^\gamma : \{\gamma \in \Gamma : a_\gamma \neq 0\} \text{ well-ordered} \right\}$

↖ spherically complete "Hahn series" "Mal'cev-Neumann field"

③ Set $k = \left(\mathbb{F}_p((t^{p\cdot\Gamma})) \cdot \mathbb{F}_p(t^\Gamma)\right)^\wedge$
$\cap\!\!\!\!\neq$
$k^{sp} = \mathbb{F}_p((t^\Gamma))$

$\sum_i t^{-r_i} \in k^{sp} \setminus k$ $\square$

<u>Rem:</u> We also have local examples, using the ring of convergent power series

$$K_n(k) := k\langle X_1, X_2, \ldots, X_n \rangle := \left\{ \begin{array}{l}\text{power series converging on some} \\ \text{polydisc of polyradius } \vec{r}\end{array} \right\} = \mathcal{O}_{\underbrace{\mathbb{B}^n(k), \vec{0}}_{\text{in rigid-analytic sense}}}$$

This ring is even Henselian!

[Datta-Smith 2016] [Hashimoto 2010]

<u>Open Q</u> Recall R of prime char. $p>0$ is <u>F-pure regular</u> or <u>very strongly F-regular</u> if $\forall\ c \in R$ avoiding $\mathrm{Min}(R)$, $\exists\, e>0$ s.t.

$$R \xrightarrow{F^e} F^e_* R \xrightarrow{F^e_*(-\cdot c)} R \text{ is pure.}$$

① Is every excellent regular domain F-pure regular? OK if local [Datta-Smith 2016]
$T_n(k)$ is F-pure regular

② $\exists$ excellent regular domain R and a sequence $\{m_e\} \subseteq \mathrm{Max}(R)$ s.t. $\bigcap_e m_e^{[p^e]} \neq 0$?

Yes for ② $\Longrightarrow$ No for ①