Singularities of generic hypersurface projections

Problem 1. Let $X \subseteq \mathbf{P}^r$ be a smooth surface.

(a) Adapt the proof that curves are birational to a nodal plane curve to show that there is a generic linear projection $\pi\colon \mathbf{P}^r \dashrightarrow \mathbf{P}^5$ such that $\pi|_X$ is an isomorphism onto its image.

(b) If one projects to $\mathbf{P}^4$ or $\mathbf{P}^3$, what singularities do you think the image of X could have?

Remark. A theorem of Severi says that the only non-degenerate smooth surface in $\mathbf{P}^5$ that projects to a smooth surface in $\mathbf{P}^4$ is the Veronese surface $\nu_2(\mathbf{P}^2) \subseteq \mathbf{P}^5$.

For the next problem, we use the following:

Fedder's Criterion. *Let k be a field of characteristic $p > 0$, and consider $f \in R = k[\![x_1, x_2, \ldots, x_n]\!]$. Then, $R/(f)$ is F-pure if and only if*

$$f^{p-1} \notin (x_1^p, x_2^p, \ldots, x_n^p).$$

In particular, if $f^{p-1} \notin (x_1^p, x_2^p, \ldots, x_n^p)$, then $R/(f)$ is weakly normal.

Problem 2. Let k be a field of characteristic $p > 0$. Using Fedder's criterion, prove the following rings are F-pure, and hence weakly normal:

(a) $k[\![x, y, z]\!]/(xyz)$.

(b) $k[\![x, y, z]\!]/(x^2 - yz^2)$, if $\operatorname{char} k \neq 2$.

(c) $k[\![s, t, u, x, y, z]\!]/(stux^2 - stuyz^2)$, if $\operatorname{char} k \neq 2$.