SEQUENTIALLY COHEN-MACAUHLAY REES MODULES

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My talk is based on the research jointly with T. N. An, N. T.
Dung and T. T. Phuong ([TPDA]). The aim of this talk is to
investigate the question of when the Rees modules associated to
arbitrary filtration of modules are sequentially Cohen-Macaulay,
which has a previous research by [CGT]. In [CGT] they gave a characterization of the
sequentially Cohen-Macaulay Rees algebras of \( \mathfrak{m} \)-primary ideal which contains a
good parameter ideal as a reduction. However their situation is quite a bit of restricted,
so we are eager to try the generalization of their results.

Let \( R \) be a Noetherian local ring with maximal ideal \( \mathfrak{m} \), \( M \neq (0) \) a finitely generated
\( R \)-module with \( d = \dim R M < \infty \). Now look at a filtration
\[
D_0 := (0) \subseteq D_1 \subseteq D_2 \subseteq \ldots \subseteq D_\ell = M
\]
of \( R \)-submodules of \( M \), which we call the dimension filtration of \( M \), if \( D_{i-1} \) is the largest
\( R \)-submodule of \( D_i \) with \( \dim R D_{i-1} < \dim R D_i \) for \( 1 \leq i \leq \ell \), here \( \dim R (0) = -\infty \) for
convention. Then we say that \( M \) is a sequentially Cohen-Macaulay
\( R \)-module, if the quotient module
\[
C_i = D_i / D_{i-1}
\]
of \( D_i \) is a Cohen-Macaulay \( R \)-module for every \( 1 \leq i \leq \ell \).

In particular, the ring \( R \) is called a sequentially Cohen-Macaulay ring, if \( \dim R < \infty \) and
\( R \) is a sequentially Cohen-Macaulay module over itself.

Let \( \mathcal{F} = \{ F_n \}_{n \in \mathbb{Z}} \) be a filtration of ideals of \( R \) such that \( F_1 \neq R \), \( M = \{ M_n \}_{n \in \mathbb{Z}} \) an
\( \mathcal{F} \)-filtration of \( R \)-submodules of \( M \). Then we put
\[
\mathcal{R} = \sum_{n \geq 0} F_n t^n \subseteq R[t], \quad \mathcal{R}' = \sum_{n \in \mathbb{Z}} F_n t^n \subseteq R[t, t^{-1}], \quad \mathcal{G} = \mathcal{R}' / t^{-1} \mathcal{R}'
\]
and call them the Rees algebra, the extended Rees algebra and the associated graded ring
of \( \mathcal{F} \), respectively. Similarly we set
\[
\mathcal{R}(M) = \sum_{n \geq 0} t^n \otimes M_n \subseteq R[t] \otimes_R M, \quad \mathcal{R}'(M) = \sum_{n \in \mathbb{Z}} t^n \otimes M_n \subseteq R[t, t^{-1}] \otimes_R M
\]
and
\[
\mathcal{G}(M) = \mathcal{R}'(M) / t^{-1} \mathcal{R}'(M)
\]
which we call the Rees module, the extended Rees module and the associated graded
module of \( M \), respectively (here \( t \) stands for an indeterminate over \( R \)). We now assume
that \( \mathcal{R} \) is Noetherian and \( \mathcal{R}(M) \) is finitely generated. We set
\[
D_i = \{ M_n \cap D_i \}_{n \in \mathbb{Z}}, \quad C_i = \{ [(M_n \cap D_i) + D_{i-1}] / D_{i-1} \}_{n \in \mathbb{Z}}.
\]
for all \( 1 \leq i \leq \ell \). Then \( D_i \) (resp. \( C_i \)) is an \( \mathcal{F} \)-filtration of \( R \)-submodules of \( D_i \) (resp.
\( C_i \)).

The author was partially supported by Grant-in-Aid for JSPS Fellows 26-126 and by JSPS Research
Fellow.
With this notation the main results of my talk are the following.

**Theorem 1.** The following conditions are equivalent.

1. \( R'(\mathcal{M}) \) is a sequentially Cohen-Macaulay \( R' \)-module.
2. \( G(\mathcal{M}) \) is a sequentially Cohen-Macaulay \( G \)-module and \( \{G(D_i)\}_{0 \leq i \leq \ell} \) is the dimension filtration of \( G(\mathcal{M}) \).

When this is the case, \( M \) is a sequentially Cohen-Macaulay \( R \)-module.

Let \( \mathfrak{M} \) be a unique graded maximal ideal of \( R \). We set
\[
a(N) = \max\{n \in \mathbb{Z} \mid [H_{\mathfrak{M}}^t(N)]_n \neq (0)\}
\]
for a finitely generated graded \( R \)-module \( N \) of dimension \( t \), and call it the \( a \)-invariant of \( N \) (see [GW, DEFINITION (3.1.4)]). Here \( \{[H_{\mathfrak{M}}^t(N)]_n\}_{n \in \mathbb{Z}} \) stands for the homogeneous components of the \( t \)-th graded local cohomology module \( H_{\mathfrak{M}}^t(N) \) of \( N \) with respect to \( \mathfrak{M} \).

**Theorem 2.** Suppose that \( M \) is a sequentially Cohen-Macaulay \( R \)-module and \( F_1 \not\subseteq p \) for every \( p \in \text{Ass}_R M \). Then the following conditions are equivalent.

1. \( R(\mathcal{M}) \) is a sequentially Cohen-Macaulay \( R \)-module.
2. \( G(\mathcal{M}) \) is a sequentially Cohen-Macaulay \( G \)-module, \( \{G(D_i)\}_{0 \leq i \leq \ell} \) is the dimension filtration of \( G(\mathcal{M}) \) and \( a(G(C_i)) < 0 \) for every \( 1 \leq i \leq \ell \).

When this is the case, \( R'(\mathcal{M}) \) is a sequentially Cohen-Macaulay \( R' \)-module.

**References**


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