Chapter 3 Recipe Book Entries

March 24, 2021

1. Linear constant coefficient homogeneous equations. Those are central to the course. They look like

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0 \tag{1}$$

We seek n linearly independent solutions.

Examples: $y'' + 3y = 0$, $y^{(3)} - 3y'' + 3y' - y = 0$.

To solve them, compute the characteristic equation

$$r^n + a_1 r^{n-1} + \cdots + a_n = 0$$

(in practice replace k-th derivative in (2) by r^k) and find its roots. There are the following cases:

(a) Real distinct simple roots $r_1,\ r_2, .., r_n$: corresponding part of the general solution looks like

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \cdots + C_n e^{r_n x}$$

(b) Repeated real root r of multiplicity k: corresponding part of the general solution has general solution

$$y = C_1 e^{rx} + C_2 x e^{rx} + \cdots + C_k x^{k-1} e^{rx}$$

(c) Pair of Complex conjugate roots $r = a \pm bi$ (a, b real) corresponding part of the general solution looks like

$$y = C_1 e^{ax}\cos(bx) + C_2 e^{ax}\sin(bx)$$

(d) Pair of Complex conjugate roots $r = a \pm bi$ (a, b real) repeated with multiplicity k: corresponding part of the general solution looks like

$$\begin{aligned} y = (A_1 e^{ax}\cos(bx) + B_1 e^{ax}\sin(bx)) + x(A_2 e^{ax}\cos(bx) + B_2 e^{ax}\sin(bx)) \\ + \cdots + x^{k-1}(A_2 e^{ax}\cos(bx) + B_2 e^{ax}\sin(bx)) \end{aligned}$$

Example: if the characteristic equation has the following roots:

- $r = 2$ (simple)
- $r = 3$ (repeated, multiplicity 3)
- $r = 4 \pm 3i$ (simple pair)
- $r = 3 \pm i$ (repeated, multiplicity 2)

then the general solution looks like

$$\begin{aligned} y = C_1 e^{2x} + C_2 e^{2x} + C_3 e^{3x} + C_4 x e^{3x} + C_5 x^2 e^{3x} \\ + C_6 e^{4x}\cos(3x) + C_7 e^{ax}\sin(3x) \\ +(A_1 e^{3x}\cos(x) + B_1 e^{3x}\sin(x)) \\ + x(A_2 e^{3x}\cos(x) + B_2 e^{3x}\sin(x)) \end{aligned}$$

2. Euler equations. They aren't covered in the main body of the text but they appeared in the homework and as such they are possible exam topics. Second order Euler equations look like

$$x^2 \frac{d^2 y}{dx^2} + ax\frac{dy}{dx} + by = 0$$

They can be solved by setting $x = e^v$ and using the chain rule to transform them into equations with independent variable v. Euler equations are among the very few non-constant coefficient linear equations we learn how to handle in this class.

3. Linear non-homogeneous equations. We have two important techniques.

   (a) Undetermined coefficients. More restricted in use, often fast, especially for high order equations. Works with equations of the form

   $$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = f(x) \tag{2}$$

   with **constant coefficients** and $f(x)$ of a **special form**: finite sum of finite products of:

   - polynomials
   - exponentials
   - $\cos(kx)$, $\sin(kx)$

   but **not** $\tan(x)$, $\sec(x)$ etc.

   Note: To apply the method you first you have to solve the associated homogeneous equation (see above) to find a complementary solution, even if you are only asked to find a particular solution. The reason is that you need to know the complementary solution to check if there is duplication.

4. Variation of parameters. Much more general method. It works for non-constant coefficient linear nonhomogeneous equations such as

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = f(x)$$

with no restriction on f. We discussed only the case $n = 2$, it becomes more cumbersome for higher order equations. To apply the method you need to know two linearly independent solutions of the associated homogeneous equation.

5. Method of reduction of order. Useful if you know one solution of a linear (possibly non-constant coefficient) equation and you are seeking another. If v is a known solution, set $y = vu$, plug into a given equation and solve an equation for u to find y.