Laplace Transform Computation Techniques

This handout should be used together with the table of Laplace transforms
`https://www.math.purdue.edu/academic/files/courses/2013spring/MA26600/LT.pdf`
and it is not meant as an exhaustive list of methods. Throughout, $f(t)$ is defined for $t \geq 0$ and $F(s) = \mathcal{L}\{f(t)\}$ is its Laplace Transform. In the "Try it with" lines, you should try to take Laplace transform of $f(t)$ and Inverse Laplace transform of $F(s)$.

Disclaimer: For pretty much all of the results listed below one needs to impose certain regularity and growth assumptions on f (for example, f generally has to be of exponential order as $t \to \infty$ and piecewise continuous or piecewise smooth), and the results hold for values of s sufficiently large. To each of the formulas you should attach the interpretation "for functions f and values of s for which the formula makes sense". For precise statements you can check out the statements in the textbook.

1 Computing the Laplace Transform Using the Definition

You can compute $\mathcal{L}\{f(t)\}$ using the definition:

Definition 1 (§7.1)**.** *If f is piecewise continuous for $t \geq 0$ its Laplace Transform is*

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt. \tag{1}$$

Computing $\mathcal{L}\{f(t)\}$ using the definition is convenient for relatively simple functions. It tends to work reasonably well for functions for which the integral in (1) can be computed either directly or via an Integration By Parts.

Try it with: $f(t) = 1$, $f(t) = e^{at}$ (where $a \in \mathbb{C}$), $f(t) = t^3$, $f(t) = u_a(t)$ (where $a \in \mathbb{R}$), $f(t) = \sin(t)$, $f(t) = \sin^2(t)$.

If $f(t)$ is a linear combination of functions whose Laplace Transforms are manageably computed one can use linearity:

Theorem 1 (Linearity, §7.1 Theorem 1)**.** *If a, b are **constants**, one has*

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

Try it with: $f(t) = \cosh(t)$, $f(t) = e^{3t} + 2t^4$

2 The Inverse Laplace Transform

Ihis class we do not see a formula for computing the Inverse Laplace Transform (it exists but it involves a complex contour integral). Instead, to compute the inverse Laplace Transform we write a given function $F(s)$ into a linear combination of simpler functions for which we can compute the inverse Laplace transform using tables. Another approach is to write a given $F(s)$ as the product of two functions with simple inverse Laplace transforms and use the Convolution property (see below). When splitting $F(s)$ into a linear combination of functions implicitly we are using the fact that the Inverse Laplace Transform is also linear: if a, b are constants then one has

$$\mathcal{L}^{-1}\{aF(s) + bG(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{G(s)\}$$

Try it with: $F(s) = \frac{1}{s-4} + s^{-3/2}$, $F(s) = \frac{s}{s-9}$

Often we need to compute $\mathcal{L}^{-1}\{F(s)\}$ with $F(s)$ a rational function. In order to use linearity one often has to first expand using partial fractions expansion.

Try it with: $F(s) = \frac{1}{s^4-16}$, $F(s) = \frac{1}{(s^2+s-6)^2}$

3 Differentiation and Integration rules

subsectionDifferentiation and integration in the t variable

Theorem 2 (Theorem 1, §7.2, Table entry No 18)**.** *For nice enough $f(t)$ defined for $t \geq 0$ one has*

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).$$

This proposition is key for solving Initial Value Problems by taking the Laplace transform on both sides. It generalizes to higher derivatives (§7.2, Corollary, Table entry No 18):

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

There is an analogous result for integration:

Theorem 3 (Theorem 2, §7.2)**.**

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\} = \frac{F(s)}{s},$$

equivalently

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)d\tau$$

This Theorem (especially (3)) is very useful for eliminating factors of s in the denominator.

Try it with: $F(s) = \frac{1}{s^2(s^2-16)}$ (you could also use partial fractions or the Convolution Theorem for this one)

3.1 Differentiation and integration in the s variable.

We can also relate differentiation in s with multiplication by $-t$:

Theorem 4 (§7.4 Theorem 2, Table entry No 19)**.** *We have*

$$\mathcal{L}\{-tf(t)\} = F'(s), \tag{2}$$

equivalently

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}.$$

Equation (2) *also implies*

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s).$$

The theorem here is very useful for computing the Laplace transform of functions of the form $f(t) = t^n g(t)$; computing $\mathcal{L}\{f(t)\}$ using the definition would require several integrations by parts to eliminate the polynomial factor.

Try it with: $f(t) = t^3\cos(2t)$, $f(t) = te^{-t}\sin^2(t)$ (also look at Theorem 6 below)

It is also useful if you are looking for $\mathcal{L}^{-1}\{F(s)\}$ and it happens that $\mathcal{L}^{-1}\{F'(s)\}$ is easier to compute.
Try it with: $F(s) = \ln\frac{s^2+1}{s^2+4}$, $F(s) = \ln(1+s^{-2})$

Theorem 5 (Theorem 3, §7.4)**.** *If* $\lim_{t\to 0^+}\frac{f(t)}{t}$ *exists and is finite, and* f *is of exponential order and piecewise continuous then*

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma)d\sigma,$$

equivalently

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = t\mathcal{L}^{-1}\left\{\int_s^\infty F(\sigma)d\sigma\right\}.$$

Useful for computing $\mathcal{L}\{f(t)/t\}$ assuming $f(t)$ has the appropriate behavior near 0: note that if f is continuous then necessarily $f(0) = 0$, to guarantee that $\lim_{t\to 0^+} \frac{f(t)}{t}$ exists and is finite.

Try it with: $f(t) = \frac{\sin(t)}{t}$ (you don't want to compute $\mathcal{L}\{\frac{\sin(t)}{t}\}$ using the definition!).

4 Product and convolution

TLaplace Transform turns convolution into multiplication:

Theorem 6 (Theorem 1, §7.4, Table entry No 16)**.** *, One has*

$$\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\},$$

equivalently

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t) \tag{3}$$

Equation (3) is often quite useful for computing inverse Laplace transforms of functions which can be written as the product of functions with conveniently computable inverse Laplace transforms. The possible disadvantage of this approach is that computing the convolution might be annoying.

Try it with: $F(s) = \frac{1}{s(s^2+4s+5)}$ (also see Theorem 3 above, $F(s) = \frac{s}{(s-3)(s^2+1)}$. For both of those you can also use partial fractions and linearity.

5 Translation

5.1 Translation in the s axis

Multiplication by an exponential in t corresponds to translation in s and vice versa:

Theorem 7 (Theorem 1, §7.3, Table entry No 14)**.** *We have*

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

and

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$$

This theorem is useful both for eliminating exponential factors in t and for computing $\mathcal{L}^{-1}\{F(s)\}$ when $\mathcal{L}^{-1}\{F(s+a)\}$ is easier to compute.

Try it with: $f(t) = e^{-t}\sin^2(t)$, $F(s) = \frac{s-1}{(s+1)^2}$.

5.2 Translation in the t axis

Translation in t (multiplied by a cutoff) corresponds to multiplication by an exponential in s

Theorem 8 (Theorem 1, §7.5, Table entry No 13)**.** *We have*

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s),$$

equivalently

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a).$$

This is useful for computing the Laplace transforms of signals with delay or signals which stop after finite time (see Lesson 39, for instance.) Also useful for computing the Inverse Laplace Transforms when a factor of e^{-as} is present.

Try it with: $F(s) = \dfrac{e^{-s} - e^{2-2s}}{s-1}$, $F(s) = \frac{se^{-s}}{s^2+\pi^2}$, $f(t) = \begin{cases} 1, & 1 \le t \le 4 \\ 0, & t < 1 \text{ or } t > 4 \end{cases}$, $f(t) = \begin{cases} \cos(\pi t), & 3 \le t \le 5 \\ 0, & t < 3 \text{ or } t > 5 \end{cases}$