Plan for Today:

1. Finish § 1.3
2. Start § 1.4

Learning Goals for the day

1. Be able to apply the Theorem of Existence and Uniqueness of solutions to specific examples
2. In case it does not, be able to tell why.
3. What is a Separable ODE? Be able to identify one when you see it in nature and solve it
4. What is an implicit solution of an ODE?

Reminders:

1. Read the textbook!
2. Office Hours today 2-3 pm

[Copied from Wednesday]

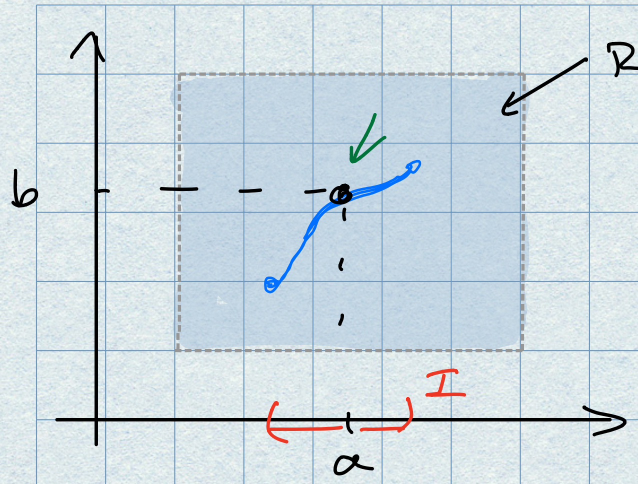
Existence & Uniqueness of Sols. Thm

S'pose: $f(x,y)$ & $D_y f := \frac{\partial f}{\partial y}(x,y)$ are continuous on a rectangle containing a pt (a,b) in its interior.

Then: There is an interval I containing a such that IVP

$$\begin{cases} \frac{dy}{dx} = f(x,y) \\ y(a) = b \end{cases}$$

has exactly one sol'n on I



Rectangle, where $f(x,y)$, $D_y f(x,y)$ cont.

Note: I need not be as wide as the rectangle!

[New Material]

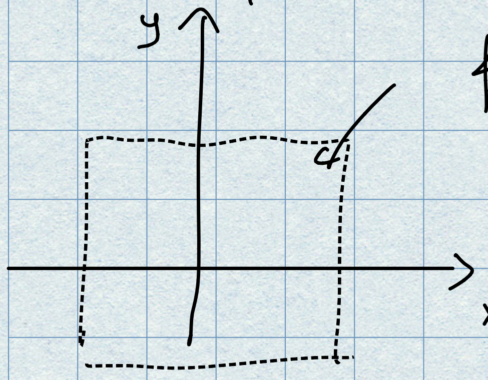
Ex: $\begin{cases} y' = \frac{4}{3} y^{\frac{1}{3}} \\ y(0) = 0 \end{cases}$

saw last time that IVP does not have unique sol'n.

$\frac{dy}{dx} = f(x,y) \qquad f(x,y) = \frac{4}{3} y^{\frac{1}{3}}$

Hope: to find rectangle containing $(0,0)$ so that f, $D_y f$ cont. there.

Not possible:



f cont for $y \geq 0$

$D_y f = \frac{4}{9} y^{-\frac{2}{3}}$ not cont. at $(x,y) = (0,0)$

Can't find rectangle that works.

Moral: f, $D_y f$ cont. is important!

Ex 2: $xy' = 3y$ ⊛

1. Check that
$$y = \begin{cases} c_1 x^3, & x \geq 0 \\ c_2 x^3, & x \leq 0 \end{cases}$$
solves ⊛ for all x and for any c_1, c_2.

For $x \geq 0$: $y = c_1 x^3$

so $xy' = 3x^2 \cdot c_1 \cdot x = 3c_1 x^3$ $=$

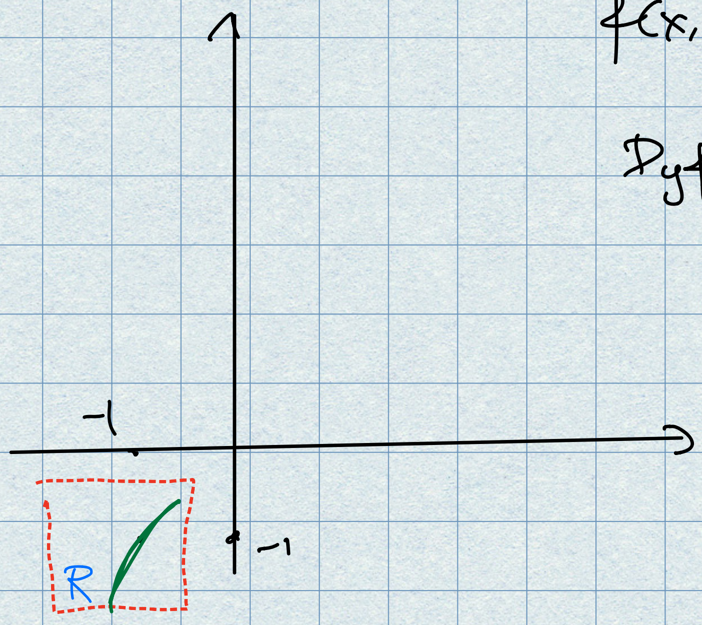
$3y = 3c_1 x^3$

so $xy' = 3y$ for $x \geq 0$

For $x \leq 0$: exercise

2. Show there is a unique sol'n of
$$\begin{cases} xy' = 3y \\ y(-1) = -1 \end{cases}$$
Want: $y' = f(x,y)$

Looking at x away from 0, so divide.
$$y' = \underbrace{\frac{3y}{x}}$$



$$f(x,y) = \frac{3y}{x}$$

$$D_y f(x,y) = \frac{3}{x}$$

in R f, $D_y f$ cont. so by thm there is exactly one sol'n in some interval containing -1.

Note: Uniqueness only holds near -1:

$$y = \begin{cases} x^3, & x \le 0 \\ Cx^3, & x \ge 0 \end{cases}$$

is a sol'n to IVP on $\mathbb{R}$ for any C: ∞ many sols on $\mathbb{R}$, only one near -1.

3. For what a does IVP

$$\begin{cases} xy' = 3y \\ y(0) = a \end{cases}$$

have a unique sol'n if any?

Can't divide by x the ODE $xy' = 3y$

when $x=0$.

or: $f(x,y)=\frac{3y}{x}$, $D_y f(x,y)=\frac{3}{x}$ not continuous near $x=0$.

Can't hope to use thm for IVP.

$$xy' = 3y \;\Big|_{y(0)=a} \text{ plug in}$$

$\to \quad 0\cdot y' = 3\cdot a \Rightarrow \quad a=0$.

So if $a\neq 0$ can't solve IVP!

If $a=0$ ∞ many sols bec. found

$$y=\begin{cases} c_1 x^3 & x\geq 0 \\ c_2 x^3 & x<0 \end{cases}$$

is a sol'n for each c_1, c_2

§1.4 Separable eq's.

Seen. $\frac{dy}{dx}=f(x)$

no y dependence

Today: $\frac{dy}{dx} = f(x)g(y)$

there is y dependence of special form: product of a fct of x & a fct of y.

Ex: $\frac{dy}{dx} = (x^2+3x)(y^2+e^y)$

$$\frac{dy}{dx} = x\, y \sin(y)$$

$$\frac{dy}{dx} = y + y x^2 = y(1+x^2)$$

Non-examples: $\frac{dy}{dx} = y + 3\sin(x)$

$$\frac{dy}{dx} = y x^3 + \ln y \cos(x)$$

Those eq's are called <u>separable</u>

<u>Ex of solving:</u>

$$\frac{dy}{dx} = y\,(x-1)$$

Divide by y: $\frac{1}{y}\frac{dy}{dx} = x-1$

separate variables: y on left, x on right.

formally "multiply both sides by dx".

$$\frac{dy}{y} = (x-1)\,dx$$

Notation that works, not that $\frac{dy}{dx}$ is a fraction.

$$\int \frac{dy}{y} = \int x-1\, dx$$

Integrate both sides

$$\Rightarrow \quad \ln|y| = \frac{x^2}{2} - x + C$$

$$\Rightarrow \quad |y| = e^{\frac{x^2}{2} - x + C}$$

$$\Rightarrow \quad |y| = \underbrace{e^{C}}_{>0} \, e^{\frac{x^2}{2} - x}$$

$$\Rightarrow \quad y = \pm e^{C} \, e^{\frac{x^2}{2} - x}$$

$$\Rightarrow \quad \boxed{y = \tilde{C} \, e^{\frac{x^2}{2} - x}}$$

$\tilde{C} \in \mathbb{R}$, replaces $\pm e^{C}$

Method for solving separable eqs:

$$\frac{dy}{dx} = k(y)\,h(x)$$

1. divide by $k(y)$ $\qquad \frac{1}{k(y)} \frac{dy}{dx} = h(x)$

2. Integrate both sides in x $\qquad \int \frac{1}{k(y)} \frac{dy}{dx}\, dx = \int h(x)\, dx$

3. substitution on left, turn into y integr. $\qquad \int \frac{1}{k(y)}\, dy = \int h(x)\, dx$

4. $\boxed{\text{Integrate!}}$ $G(y) = H(x) + C$, where ⊛

G is antider. for $\frac{1}{k(y)}$

H " for $h(x)$

Rmk: ⊛ doesn't always give $y = \ldots$

but it gives an eq'n which

y satisfies and there are no derivatives.