Plan for today:

1. Finish § 1.4
2. Start § 1.5

Learning Goals:

From § 1.4, in addition to the ones from Lesson 6

1. Be able to solve problems involving population growth, radiocarbon dating and heating/cooling

From § 1.5:

1. Be able to recognize a linear 1st order ODE when you see it, understand how it differs from types of ODEs we have studied so far.
2. Given a linear 1st order ODE, be able to compute an integrating factor for it
3. Be able to find the general solution of a linear 1st order ODE using an integrating factor
4. Be able to identify the largest interval on which an IVP for a linear 1st order ODE has a unique solution (questions of this type appear often in the finals)

Reminders:

1. Read the textbook!
2. Fill out survey (link will be emailed shortly after class).

Population Growth (application of separable eqs)

Simple model: Population where rate of births is const. in time, rate of deaths is const. in time.

$P(t) \to$ population at time t.

$\beta \to$ births/person in unit time

$\delta \to$ deaths/person in unit time

How does $P(t)$ evolve? In time Δt:

$$\Delta P = \underbrace{\beta \cdot P \cdot \Delta t}_{\text{births}} - \underbrace{\delta \cdot P \cdot \Delta t}_{\text{deaths}}$$

$$= (\beta - \delta) P \Delta t$$

separable!

$$\lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t} = \underbrace{\underbrace{P'(t)}_{?} = \overbrace{(\beta-\delta)}^{\text{const.}} \underbrace{P(t)}_{?}} \qquad \boxed{y' = ky}$$

$$\frac{dP}{dt} = (\beta - \delta)P \Rightarrow \int \frac{dP}{P} = \int (\beta-\delta)\,dt$$

$$\Rightarrow \ln P = (\beta-\delta)t + C$$

$$\Rightarrow \quad P = \underbrace{e^{C}} e^{(\beta-\delta)t}$$

$$\Rightarrow \boxed{P(t) = \underbrace{P_0}_{} e^{(\beta-\delta)t}}$$

Population at time $t=0$.

Note:

$$\lim_{t\to\infty} P(t) = \begin{cases} \infty & , \beta > \delta \\ P_0 & , \beta = \delta \\ 0 & , \beta < \delta \end{cases}$$

Ex: Culture of bacteria s.t. their number increased 3-fold in 10 hours. How long did it take for # of bacteria to double (assuming growth model above).

Have: $P(t) = P_0 e^{kt}$ $(k = \beta - \delta)$

$\hookrightarrow$ t in hours.

$P(10) = 3P_0$ (increased 3-fold)

$\cancel{P_0} e^{k \cdot 10} = 3\cancel{P_0}$

$\Rightarrow k \cdot 10 = \ln 3 \Rightarrow k = \frac{\ln 3}{10}$

Looking for: t_1: $P(t_1) = 2P_0$

$\cancel{P_0} e^{kt_1} = 2\cancel{P_0} \Rightarrow e^{\frac{\ln 3}{10} t_1} = 2$

$$\Rightarrow \quad \frac{\ln 3}{10} t_1 = \ln 2 \Rightarrow t_1 = \frac{\ln 2}{\ln 3} \cdot 10$$
hours.

Linear 1st Order ODE (1.5)

Seen: $\frac{dy}{dx} = f(x)$ (*)

$\frac{dy}{dx} = f(x) g(y)$ (separable)

Today: $\frac{dy}{dx} + \underbrace{P(x)}_{\text{no } y \text{ dependence.}} y = \underbrace{Q(x)}_{\text{no } y \text{ dependence}}$

Ex: $\frac{dy}{dx} + x^2 y = 3\sin(x)$

$\frac{dy}{dx} = 3\cos(x) y - x^2$

Non-ex: $\frac{dy}{dx} - 3xy^2 = \cos(x)$ squared

$\frac{dy}{dx} = 3xy + y^5$ additional term involving y.

Note: $\frac{dy}{dx} = 3xy$ both linear & separable

$\frac{dy}{dx} = \cos(x)$ both linear & (*)

How to solve:

Let $\quad p(x) = e^{\int P(x)dx}$

Observation:

$$\left(p(x)y(x)\right)' = p'(x)y(x) + p(x)y'(x)$$

$$= P(x)e^{\int P(x)dx}y + e^{\int P(x)dx}y'$$

$$= p(x)\left(P(x)y + y'\right) \quad (*)$$

So given

$$\frac{dy}{dx} + P(x)y = Q(x):$$

Multiply through by $p(x) = e^{\int P(x)dx}$. By (*)

$$p(x)\left(\frac{dy}{dx} + P(x)y\right) = p(x)Q(x)$$

$$\left(p(x)y(x)\right)' = p(x)Q(x)$$

Eq'n of type (*)

$$p(x)y(x) = \int p(x)Q(x)dx + C$$

$$\Rightarrow \quad \boxed{y(x) = \frac{1}{p(x)}\left(\int p(x)Q(x)dx + C\right)}$$

General sol'n to $\frac{dy}{dx} + P(x)y = Q(x)$

$p(x) = e^{\int P(x)dx}$ is called an <u>integrating factor</u>

(More generally an integrating factor is a fct you multiply both sides of an eqn by and it turns them into derivatives)

Ex 1: $\frac{dy}{dx} + \underbrace{2x}_{P(x)} y = \underbrace{x}_{Q(x)}$

the coefficient of y

Integrating factor: $p(x) = e^{\int P(x)\,dx} = e^{\int 2x\,dx} = e^{x^2}$

So: $e^{x^2}\frac{dy}{dx} + \underbrace{e^{x^2}\cdot 2x}_{\frac{d}{dx}(e^{x^2})} y = e^{x^2}\cdot x$

$\Rightarrow \frac{d}{dx}\left(e^{x^2}y\right) = xe^{x^2}$

$\Rightarrow e^{x^2}y = \int x e^{x^2}\,dx + C$

$\Rightarrow e^{x^2}y = \frac{1}{2}e^{x^2} + C$

$\Rightarrow y = \frac{1}{2} + Ce^{-x^2}$ ← General soln

Ex 2: $x\frac{dy}{dx} = x^3 y + \cos(x)$ (*)

Integrating factor!

$y' + P(x)y = Q(x)$

to bring (*) into standard form:

$$\frac{dy}{dx} \underbrace{-x^2}_{P(x)} y = \frac{\cos(x)}{x}$$

$$p(x) = e^{\int P(x)\,dx} = e^{-\frac{x^3}{3}}$$

Finish example.