Plan for today:

Finish § 1.5

Learning goals:

1. Existence and uniqueness of solutions for IVP involving linear 1st order equations. In particular, be able to find the largest interval where a unique solution exists (popular question in finals)
2. Be able to set up a differential equation for mixing problems and solve it, both when the volume of the liquid in the container is constant and when it changes with time. (Also popular in finals)
3. When prompted, be able to solve a differential equation of the form dy/dx=f(x,y) by regarding y as the independent variable instead of x

Reminders:

1. Ungraded mock quiz on Monday
2. OH today 2-3 pm
3. Read the textbook!

Last time: $y' + P(x)y = Q(x)$ linear 1st order

To solve: $\rho = e^{\int P(x)dx}$ (integrating factor)

found: $\rho y = \int \rho(x) Q(x) dx + C$

If initial cond's are given we can find c.

always $\neq 0$

Thm: Existence & uniqueness of sols for 1st order linear ODE.

Suppose: $P(x)$, $Q(x)$ cont. on interval I, which contains x_0. Then the IVP

$$\begin{cases} \frac{dy}{dx} + P(x)y = Q(x) \\ y(x_0) = y_0 \end{cases}$$

assume more structure, get stronger result

has a unique solution on all of I.

This sol'n is given by

don't memorize, follow the procedure to derive it every time.

$$\left[\rho(x)\, y(x) = \int_{x_0}^{x} \rho(t)\, Q(t)\,dt + y_0 \right. \qquad \text{with } \rho(x) = e^{\int_{x_0}^{x} P(t)\,dt} \quad (*)$$

<u>Ex:</u> check that Theorem from Lesson 5 (§1.3) also applies for linear ODE

$(*)$ gives all sols of $y' + Py = Q$ for some x_0, y_0, i.e. there are no singular sols, $(*)$ is <u>the</u> general sol'n.

<u>Ex:</u> Find largest interval on which there exists a unique sol'n to

$$\begin{cases} (x+1)y' + \ln(x+4)\,y = e^x \\ y(1) = 2 \end{cases} \qquad (*)$$

Want: $\quad 1 \cdot y' + P(x)\,y = Q(x)$

divide by $x+1$

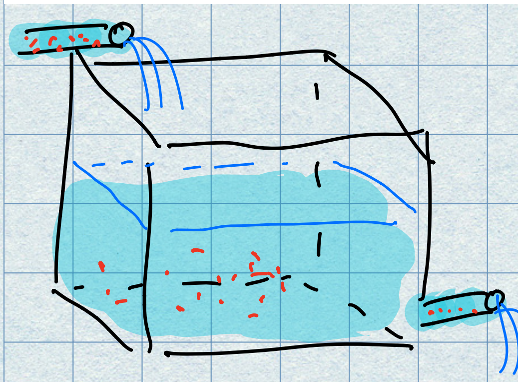
$$y' + \underbrace{\frac{\ln(x+4)}{x+1}}_{P(x)}\, y = \underbrace{\frac{e^x}{x+1}}_{Q(x)}$$

What is the largest interval containing 1 so that P, Q are cont. in it?

<u>Ans:</u> $I = (-1, \infty)$, so by thm there is a unique sol'n to IVP $(*)$ on $(-1,\infty)$

(Integrating factor would be $e^{\int \frac{\ln(x+4)}{x+1}dx}$)

34. Consider a reservoir with a volume of 8 billion cubic feet (ft^3) and an initial pollutant concentration of 0.25%. There is a daily inflow of 500 million ft^3 of water with a pollutant concentration of 0.05% and an equal daily outflow of the well-mixed water in the reservoir. How long will it take to reduce the pollutant concentration in the reservoir to 0.10%?



$V = 8{,}000$ million ft^3

$r_{in} = 500$ million ft^3/day ↑ rate

$c_{in} = 0.05\%$

$r_{out} = 500$ million ft^3/day

Set up a dif. eq'n for quantity (in million ft^3) of pollutant in reservoir: call it $q(t)$. ↑ time in days.

How does it change in time Δt?

$$\Delta q = \underbrace{c_{in}\, r_{in}\, \Delta t}_{\text{incoming}} - \underbrace{c_{out}\, r_{out} \cdot \Delta t}_{\text{outgoing}}$$

$$= c_{in}\, r_{in}\, \Delta t - \underbrace{\frac{q(t)}{V}}_{\text{percentage}}\, r_{out}\, \Delta t$$

$$= \overset{0.05\%}{\overline{0.0005}} \cdot 500 \frac{m\,ft^3}{\text{day}} \Delta t\, \text{day} - \frac{q(t)}{8000} \frac{m\,ft^3}{m\,ft^3} \cdot 500 \frac{m\,ft^3}{(\text{day})} \Delta t\, \text{day}$$

So: $\boxed{\dfrac{dq}{dt} = 0.25 - \dfrac{1}{16}q(t)}$ linear 1st order ODE.

$$\frac{dq}{dt} + \frac{1}{16}q(t) = 0.25 \qquad p(t) = e^{\int \frac{1}{16}dt} = e^{\frac{t}{16}}$$

$$\left(q(t)\,p(t)\right)' = 0.25\,p(t)$$

$$q(t)\,p(t) = \int 0.25\,p(t)\,dt + C$$

$$e^{t/16}q(t) = \int 0.25\,e^{t/16}\,dt + C$$

Use that $\dfrac{q(0)}{V} = 0.0025$ to compute C.

Finish $\boxed{\text{Ans: } \sim 22.2 \text{ days}}$

Rmk: Rate in = Rate out so volume in container stays constant.

What if it doesn't?

Modify example: Everything else same, change:

$$r_{in} = 500 \text{ m ft}^3/\text{day}$$

$$r_{out} = 700 \text{ m ft}^3/\text{day}.$$

Volume of water depends on t!

$$V(0) = 8000 \text{ m ft}^3$$

$V(t) = 8000 + 500t - 700t$ ← volume at time t

$= 8000 - 200t.$

Changes ODE! $\quad c_{out} = \dfrac{q(t)}{V(t)} = \dfrac{q(t)}{8000-200t}$

So:

$$\frac{dq}{dt} = \overbrace{0.25}^{c_{in}\cdot r_{in}} - \overbrace{700}^{r_{out}} \overbrace{\frac{q(t)}{8000-200t}}^{c_{out}}$$

Still linear 1st order ODE, more complicated.

Exercise: finish; solution follows below.

$$\frac{dq}{dt} + \frac{7}{80-2t}\, q = 0.25$$

Integrating factor: $\rho = e^{\int \frac{7}{80-2t} dt} = e^{-\frac{7}{2}\ln(80-2t)}$

$$= \frac{1}{(80-2t)^{7/2}}$$

$$\frac{1}{(80-2t)^{7/2}} \frac{dq}{dt} + \left(\frac{1}{(80-2t)^{7/2}}\right)' q = \frac{0.25}{(80-2t)^{7/2}}$$

$$\left(\frac{1}{(80-2t)^{7/2}}\, q\right)' = \frac{0.25}{(80-2t)^{7/2}}$$

$$\frac{1}{(80-2t)^{7/2}}\, q = \int \frac{0.25}{(80-2t)^{7/2}} dt + C$$

$$\frac{1}{(80-2t)^{7/2}}\, q(t) = 0.25 \cdot \frac{1}{5}\, (80-2t)^{-\frac{5}{2}} + C$$

So $q(t) = \frac{1}{20}(80-2t) + C(80-2t)^{7/2}$

Since $q(0) = 0.0025 \cdot 8000 = 20$

$$20 = \frac{1}{20} \cdot 80 + C\, 80^{7/2}$$

$$\Rightarrow \quad C = \frac{16}{80^{7/2}}$$

So:

$$q(t) = \frac{1}{20}(80-2t) + \frac{16}{80^{7/2}}(80-2t)^{7/2}$$

To find when concentration becomes 0.10 %

$$\frac{q(t)}{V(t)} = 0.001 \Rightarrow$$

$$\Rightarrow \frac{1}{20} \frac{80-2t}{8000-200t} + \frac{16}{80^{7/2}} \frac{(80-2t)^{7/2}}{(8000-200t)} = 0.001$$

$$\Rightarrow \frac{1}{20} + \frac{16}{80^{7/2}}(80-2t)^{5/2} = 0.1$$

$$\Rightarrow (80-2t)^{5/2} = 0.05 \cdot \frac{80^{7/2}}{16}$$

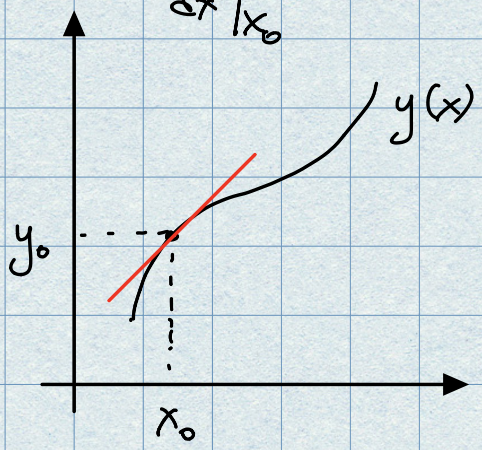
$$\Rightarrow 80-2t = \left(0.05 \frac{80^{7/2}}{16}\right)^{\frac{2}{5}}$$

$$\Rightarrow t = \frac{1}{2}\left(80 - \left(0.05 \frac{80^{7/2}}{16}\right)^{\frac{2}{5}}\right)$$

$$= 17.026$$

Hint for Problem 27:

If $\left.\frac{dy}{dx}\right|_{x_0} \neq 0$ then near x_0 we can solve for x and find x as a function of y w/ derivative



$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

Sometimes given ODE

$$\frac{dy}{dx} = f(x,y)$$

it is convenient to solve $\frac{dx}{dy} = \frac{1}{f(x,y)}$ w/

y the independent variable, if $f(x,y) \neq 0$.