Plan for today:
Start 1.6

Learning goals for the day:
1. Get used to scouting for useful substitutions: typical examples include equations of the form $y' = F(ax+by)$, $y' = F(y/x)$ (homogeneous equations)
2. Be able to recognize Bernoulli equations (and solve them using a substitution)
3. Realize that when you perform a substitution your final answer should still be in terms of the original dependent and independent variables.

Reminders:
1. HW due tomorrow on Gradescope and MyLab Math
2. Ungraded Quiz 0 available on Gradescope 9.30 am today-9.30 am tomorrow.
3. Read the textbook!
4. OH today at 1pm

1.6 Substitutions & Exact Eq's

$$\frac{dy}{dx} = F(\underbrace{a(x,y)}_{\text{some convenient expression}})$$

hope: setting $v = a(x,y)$ can reach $\frac{dv}{dx} = \tilde{F}(x,v)$ that we can solve

Ex: $\frac{dy}{dx} = (\underbrace{y - x + 3}_{= v})^2$ (*) ← not linear, nor separable.

Want: dif. eq' for v.

$v = y - x + 3 \Rightarrow y = v + x - 3$ ← take $\frac{d}{dx}$ der. on both sides

so: $\frac{dy}{dx} = \frac{dv}{dx} + 1$

(*) $\Rightarrow \frac{dv}{dx} + 1 = v^2 \Rightarrow \frac{dv}{dx} = v^2 - 1$ ← separable!

$$\frac{dv}{v^2 - 1} = dx \rightarrow \int \frac{dv}{v^2 - 1} = \int dx$$

$$\Rightarrow \int \frac{dv}{(v-1)(v+1)} = \int dx$$

Partial fractions: $\frac{A}{v-1} + \frac{B}{v+1} = \frac{1}{(v-1)(v+1)}$

$$\Rightarrow \begin{cases} A = \frac{1}{2} \\ B = -\frac{1}{2} \end{cases}$$

So: $\int \frac{1}{2}\frac{1}{v-1} - \frac{1}{2}\frac{1}{v+1}\, dv = \int dx$

$$\frac{1}{2}\ln(v-1) - \frac{1}{2}\ln(v+1) = x + C$$

Found v implicitly, looking for y! So plug in back: $v = y - x + 3$

So: $\frac{1}{2}\ln(y-x+2) - \frac{1}{2}\ln(y-x+4) = x + C$

implicit general sol'n for $\circledast$

In general: $\frac{dy}{dx} = F(\underbrace{ax+by+c}_{\text{linear eq'n } x,y})$, substitute

$v = ax+by+c \longrightarrow$ separable eq'n for v.

Homogeneous eq's:

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

Ex: $x^2\frac{dy}{dx} = xy + x^2 e^{y/x}$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + e^{y/x}$$

Substitution: $v = \frac{y}{x}$

Solve for y: $y = xv$. Then: $\frac{dy}{dx} = x\frac{dv}{dx} + v$ (pr. rule)

Find: $x\frac{dv}{dx} + \cancel{v} = \cancel{v} + e^v$

$$x\frac{dv}{dx} = e^v \quad \text{separable!}$$

$$\Rightarrow \frac{dv}{e^v} = \frac{dx}{x} \Rightarrow -e^{-v} = \ln x + C$$

substitute:

$$-e^{-y/x} = \ln x + C.$$

In general:

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right), \text{ set } v = \frac{y}{x}.$$

Then $y = xv$,

$$x\frac{dv}{dx} + v = F(v) \Rightarrow \underbrace{\frac{dv}{dx} = \frac{F(v)-v}{x}}_{\text{separable.}}$$

3. Bernoulli:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \quad \text{for some } n$$

Note: if $n=0$: $\frac{dy}{dx} + P(x)y = Q(x)$ linear!

$n=1$: $\frac{dy}{dx} + P(x)y = Q(x)y$

$$\frac{dy}{dx} = (Q(x) - P(x))y \quad \text{separable!}$$

Solve by substitution $v = y^{1-n}$.

Ex:

$$x^2 y' + 2xy = 5y^4$$

divide $y' + \frac{2}{x} y = \frac{5}{x^2} y^4$

Let $v = y^{1-4} = y^{-3} \Rightarrow y = v^{-\frac{1}{3}}$. Now $\frac{dy}{dx} = -\frac{1}{3} v^{-\frac{4}{3}} \frac{dv}{dx}$

Substitute:

$$-\frac{1}{3} v^{-\frac{4}{3}} \frac{dv}{dx} + \frac{2}{x} v^{-\frac{1}{3}} = \frac{5}{x^2} \left(v^{-\frac{1}{3}}\right)^4$$

Multiply through by $v^{\frac{4}{3}}$

$$-\frac{1}{3} \frac{dv}{dx} + \frac{2}{x} v = \frac{5}{x^2}$$

$$\frac{dv}{dx} - \frac{6}{x} v = -\frac{15}{x^2}$$ linear eq'n!

Integrating factor: $\rho(x) = e^{\int -\frac{6}{x} dx} \ldots$ $*$

Exercise: finish this. Sol'n at the end.

In HW: show that Bernoulli $\longrightarrow$ linear w/ subst. $v = y^{1-n}$

$\underline{*}$ $x \frac{1}{y} y' + \ln(x) \ln(y) = (\ln y)^4 e^x$ (hard)

Read Ex. 6 in book and then try it.

Warning: do not try to compute the last integral explicitly!

Aside 1: How we found that $v = y^{1-n}$ works:

$$\frac{dy}{dx} + P(x)\, y = Q(x)\, y^n$$

Try $v = y^{\alpha} \Rightarrow y = v^{\frac{1}{\alpha}}$ for unknown α

$$\frac{dy}{dx} = \frac{1}{\alpha} v^{\frac{1}{\alpha}-1} \frac{dv}{dx}$$

$$\frac{1}{\alpha} v^{\frac{1}{\alpha}-1} \frac{dv}{dx} + P(x)\, v^{\frac{1}{\alpha}} = Q(x)\, v^{\frac{n}{\alpha}}$$

$$\frac{1}{\alpha} \frac{dv}{dx} + P(x)\, \underbrace{v^{\frac{1}{\alpha}}\, v^{1-\frac{1}{\alpha}}}_{v} = Q(x)\, \cancel{v^{\frac{n}{\alpha}+1-\frac{1}{\alpha}}}$$

if

$$\frac{n}{\alpha} + 1 - \frac{1}{\alpha} = 0$$

$$n + \alpha - 1 = 0$$

$$\alpha = 1 - n$$

so: $v = y^{1-n}$

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Aside 2: Solving the hard problem:

$$x \frac{1}{y} y' + \ln(x) \ln(y) = (\ln y)^4 e^x$$

Note: $\frac{1}{y} y' = (\ln y)'$. Try $v = \ln y$.

Find:

$$x v' + \ln(x)\, v = v^4 e^x$$

$$\Rightarrow v' + \frac{\ln(x)}{x} v = v^4 \frac{e^x}{x}$$

Bernoulli!

Set $u = v^{1-4} = v^{-3} \Rightarrow v = u^{-\frac{1}{3}}$

$$\frac{dv}{dx} = -\frac{1}{3} u^{-\frac{4}{3}} \frac{du}{dx}$$

$$-\frac{1}{3} u^{-\frac{4}{3}} \frac{du}{dx} + \frac{\ln(x)}{x} u^{-\frac{1}{3}} = u^{-\frac{4}{3}} \frac{e^x}{x}$$

$$\frac{du}{dx} - 3\frac{\ln(x)}{x} u = -3\frac{e^x}{x} \quad \text{linear!}$$

Integrating factor: $\quad p(x) = e^{-3\int \frac{\ln(x)}{x} dx} = e^{-\frac{3}{2}(\ln x)^2}$

So: $\quad \frac{d}{dx}\left(e^{-\frac{3}{2}(\ln x)^2} u\right) = -3\frac{e^x}{x} e^{-\frac{3}{2}(\ln x)^2}$

$$\Rightarrow e^{-\frac{3}{2}(\ln x)^2} u = \int -3\frac{e^x}{x} e^{-\frac{3}{2}(\ln x)^2} dx + C$$

$u = v^{-3}$

$$\Rightarrow v^{-3} = e^{-\frac{3}{2}(\ln x)^2}\left(\int -3\frac{e^x}{x} e^{-\frac{3}{2}(\ln x)^2} dx + C\right)$$

$v = \ln y$

$$\Rightarrow (\ln y)^{-3} = e^{-\frac{3}{2}(\ln x)^2}\left(\int -3\frac{e^x}{x} e^{-\frac{3}{2}(\ln x)^2} dx + C\right)$$

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Aside 3: Solving the Bernoulli example.

Pick up from $*$:

$$\frac{dv}{dx} - \frac{6}{x} v = -\frac{15}{x^2} \quad \text{linear eq'n!}$$

Integrating factor: $\quad p(x) = e^{\int -\frac{6}{x} dx} = e^{-6\ln x} = \frac{1}{x^6}$

So $$\left(\frac{1}{x^6} v\right)' = -\frac{15}{x^8}$$

$$\Rightarrow \frac{1}{x^6} v = \frac{15}{7} x^{-7} + C$$

$$\Rightarrow v = \frac{15}{7} x^{-1} + Cx^{-6}$$

$$\Rightarrow y^{-3} = \frac{15}{7} x^{-1} + Cx^{-6} \qquad //$$