Plan for today:

§ 1.6

Learning goals for the day:

1. Get used to looking for exactness (includes knowing the test for checking if an equation is exact). If an equation is given in the form M(x,y)dx+N(x,y)dy exactness should be your first thought.

Reminders

1. Quiz 1 will become available at 9.30 am tomorrow and stay available for 24 hours.
2. Solutions to Quiz 0 posted on the course calendar.
3. Read the textbook!

$x^2+y^2 = C$

Exact ODE

Recall: implicit sols: Eq'n $F(x,y) = C$ satisfied by a soln of a given ODE.

Sup. given $F(x,y(x)) = C$ for some fct $y(x)$.

Take ∂_x der.

$$\partial_x \left(F(x,y(x)) \right) = 0$$

chain rule $\Rightarrow$ $(\partial_x F)(x,y(x)) + \partial_y F(x,y(x)) \frac{dy}{dx} = 0$

so: $y(x)$ is a sol'n of dif eq'n

$$\frac{dy}{dx} \partial_y F + \partial_x F = 0.$$

What if we go the other way, and we are given

$$\frac{dy}{dx} N(x,y) + M(x,y) = 0 \quad ? \qquad (*)$$

If we can arrange that $N = \partial_y F$ for some F, $M = \partial_x F$ for the same F, then $F(x,y) = C$ will be a general sol'n for $(*)$.

Qns: Given (*), when can we find such F? how?

If $M = \partial_x F \Rightarrow \partial_y M = \partial_{xy} F$
$N = \partial_y F \Rightarrow \partial_x N = \partial_{yx} F$ } if F nice $\boxed{\partial_x N = \partial_y M}$

So: Given M, N can't always find F (need ↑)

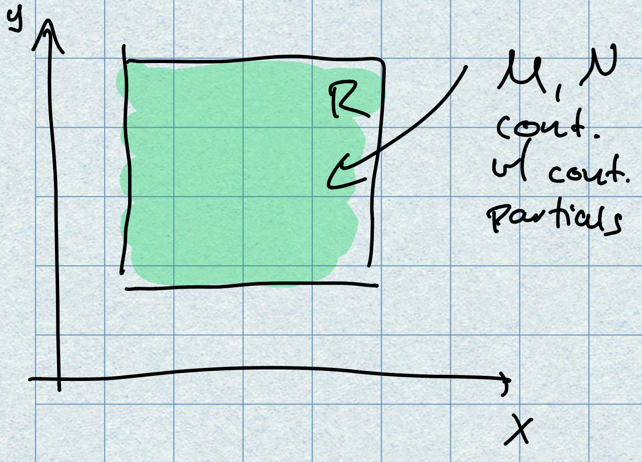
Def'n The differential form of ODE (*) is $M(x,y)\,dx + N(x,y)\,dy = 0$.
(Notation)

Def'n: An ODE in differential form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is called exact when there is an F so that $M = \partial_x F$, $N = \partial_y F$.

Thm: Let $M(x,y)$, $N(x,y)$ be cont, w/ cont. partial derivatives on a rectangle $R = [a,b] \times [c,d]$.



Then

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ on } R$$

Ex: $\underbrace{3y}_{M}\,dx + \underbrace{y^2}_{N}\,dy = 0$

Is this exact?

$$\left.\begin{aligned} \partial_y M &= 3 \\ \partial_x N &= 0 \end{aligned}\right\} \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}, \text{ so not exact.}$$

Rmk: not saying that we can't solve the eq'n, just that there is no F with $\partial_x F = 3y$, $\partial_y F = y^2$.

$$3y\,dx + y^2\,dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{3}{y} \quad \text{separable.}$$

Ex 2: $$\left(x + \arctan(y)\right) dx + \frac{x+y}{1+y^2}\,dy = 0 \quad (*)$$

$$M = x + \arctan(y), \quad N = \frac{x+y}{1+y^2}$$

$$\partial_y M = \frac{1}{1+y^2} \qquad \partial_x N = \frac{1}{1+y^2}$$

the same on $\mathbb{R}^2$ (so in any rectangle)

$\Rightarrow (*)$ is exact.

How do we find $F(x,y)$ so that $M = \partial_x F$, $N = \partial_y F$?

$$\partial_x F = M = x + \arctan(y)$$

$\Longrightarrow$ $F(x,y) = \int x + \arctan(y)\,dx$

integrate in x

$$F(x,y) = \frac{x^2}{2} + x\arctan(y) + \underbrace{g(y)}_{\text{"constant" of integration}}$$

∂_x

$N = \partial_y F \Rightarrow \frac{x+y}{1+y^2} = \frac{x}{1+y^2} + \frac{y}{1+y^2} = \partial_y F = \frac{x}{1+y^2} + g'(y)$

$$\Rightarrow \underbrace{g'(y) = \frac{y}{1+y^2}}_{} \Rightarrow g(y) = \frac{1}{2}\ln(1+y^2) + C$$

if you find $g'(y)$ depending on x it means you made a mistake, or eq'n not exact.

<u>So:</u> $F(x,y) = 0 \Leftrightarrow \boxed{\frac{x^2}{2} + x\arctan(y) + \frac{1}{2}\ln(1+y^2) + C = 0}$

is a general sol'n.

<u>Recap:</u> $M(x,y)dx + N(x,y)dy = 0$

1st: check if $\partial_y M = \partial_x N$.

If yes, set $\partial_x F = M$, $\partial_y F = N$ for F tbd.

Integrate $\partial_x F = M$ in x, find F depends on unknown $g(y)$, differentiate in y, set equal to N.

Then $F(x,y) = C$ is a gen. sol'n in implicit form.

Example Homog. eqn:

$$(x^2-y^2)\,y' = 2xy$$

Hope that we can write as $y' = F\left(\frac{y}{x}\right)$

$$y' = \frac{2xy}{x^2-y^2}$$

Divide by x^2 top & bottom: (assuming $x \neq 0$)

$$y' = \frac{2\,y/x}{1-\left(\frac{y}{x}\right)^2} \quad \text{so homog.}$$

Set $v = \frac{y}{x}$: $\frac{dy}{dx} = x\frac{dv}{dx} + v$

$$x\frac{dv}{dx} + v = \frac{2v}{1-v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v - v + v^3}{1-v^2} = \frac{v(1+v^2)}{1-v^2}$$

$$\Rightarrow \int \frac{1-v^2}{v(1+v^2)}\,dv = \int \frac{dx}{x} \qquad (*)$$

Partial fr: $\frac{A}{v} + \frac{Bv+C}{1+v^2} = \frac{1-v^2}{v(1+v^2)}$

$$\Rightarrow A + Av^2 + Bv^2 + Cv = 1 - v^2$$

$$\Rightarrow \begin{cases} A+B=-1 \\ A=1 \\ C=0 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=-2 \\ C=0 \end{cases}$$

So: $(\ast)$ $$\int \frac{1}{v} - \frac{2v}{1+v^2}\, dv = \int \frac{dx}{x}$$

$$\Rightarrow \ln|v| - \ln|1+v^2| = \ln|x| + C$$

$$\Rightarrow \ln\left|\frac{y}{x}\right| - \ln\left|1+\frac{y}{x}\right| = \ln|x| + C$$

$$\Rightarrow \ln\left|\frac{\frac{y}{x}}{1+\frac{y}{x}}\right| = \ln|x| + \ln\tilde{C} \qquad \tilde{C} > 0$$

$$\Rightarrow \frac{y}{x+y} = \pm\tilde{C}x = Cx \qquad C \in \mathbb{R}$$

$$\Rightarrow y(1-cx) = cx^2$$

$$\Rightarrow y = \frac{cx^2}{1-cx}$$