Plan for today:
Finish § 1.6
Start § 2.1

Learning Goals:

1. Be able to solve reducible 2nd order equations. Those are among the very few cases of possibly non-linear 2nd order equations we will see in this class. Keep them in mind because they appear on the finals.
2. Be able to set up and solve a differential equation describing the evolution of a population

Reminders-announcements

1. 1 hour left to submit Quiz 1!
2. Homework hint: One of the homework problems asks you to find the value of a variable numerically. This means that you need to use a computer algebra system to find the value of the variable, do not try to solve for it analytically.

Reducible 2nd order eq's.

General form: $F(y'', y', y, x) = 0$ (*)

Sometimes simpler form allows us to reduce (*) to a 1st order eq'n.

1: y missing

Ex: $xy'' + y' = 4x$ (*) (note: y not present)

Set: $v = y'$

Now $y'' = v'$

(*) $\underbrace{x v' + v}_{\text{linear!}} = 4x$ 1st order eq'n!

(in general may or may not be linear)

$$\frac{d}{dx}(xv) = 4x$$

$$\Rightarrow xv = \int 4x\,dx \Rightarrow xv = 2x^2 + C_1$$

$$\Rightarrow \quad v = 2x + \frac{c_1}{x} \qquad (x \neq 0)$$

<u>Find y!</u>

$$y' = 2x + \frac{c_1}{x} \quad \Rightarrow \quad y(x) = x^2 + c_1 \ln|x| + c_2$$

2 constants, as expected for 2nd order.

<u>Case 2:</u> x is missing from ODE.

<u>Ex:</u> $\quad y'' = 2yy' \quad (\ast) \qquad$ (x not appearing)

think of y as the independent variable

$$v = \frac{dy}{dx}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx} v \overset{\text{chain}}{\underset{\text{rule}}{=}} \frac{dv}{dy}\frac{dy}{dx}$$

$$= \frac{dv}{dy} v$$

Chain Rule

$$\left(f(g(x))\right)' = f'(g(x))\, g'(x)$$

$z = f(y)$

$y = g(x)$

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$$

$$(\ast) \quad \frac{dv}{dy} v = 2yv \qquad (v \neq 0)$$

$$\Rightarrow \quad \frac{dv}{dy} = 2y$$

$$\Rightarrow \quad v = y^2 + C_1 \Rightarrow \frac{dy}{dx} = y^2 + C_1$$

<u>Find y:</u> Exercise, take cases for $C_1 > 0$, < 0, $= 0$

Sol'n at the end.

2.1 Seen: Population growth model assuming constant birth & death rate.

$$\frac{dP}{dt} = (\beta - \delta) P = kP$$

β: births / unit time / unit of population

δ: death/...

$$P(t) = Ce^{kt} = Ce^{(\beta-\delta)t}$$

More generally: birth rate, death rate might depend on P, t

$$\frac{dP}{dt} = \Big(\underbrace{\beta(t,P)}_{\text{birth rate}} - \underbrace{\delta(t,P)}_{\text{death rate}}\Big) P \quad (*)$$

Logistic Equation: Based on assumption that birth rate decreases linearly as population increases, death rate const.

$$\beta = \underbrace{\beta_0}_{} - \underbrace{\beta_1}_{} P$$

positive const.

Plug in:

$(*) \rightarrow$
$$\frac{dP}{dt} = \Big((\beta_0 - \beta_1 P) - \delta \Big) P$$

$$= \underbrace{(\beta_0 - \delta)}_{a} P - \underbrace{\beta_1}_{b} P^2 \quad (**)$$

If $a, b > 0$ (**) called a logistic equation

$$\frac{dP}{dt} = aP - bP^2 \Rightarrow \boxed{\frac{dP}{dt} = kP(M-P)} \qquad M = \frac{a}{b}, \quad k = b$$

Rmks: a) Only dependent variable P present on RHS of logistic eqn. Such ODEs are called autonomous

b) RHS $= 0$ when $P = 0$, $P = M$: $P(t) \equiv 0$, $P(t) \equiv M$ are both solutions (equilibrium sols).

Solve! Separable:

$$\int \frac{dP}{P(M-P)} = \int k\, dt$$

partial $\Rightarrow$ fractions

$$\int \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right) dP = kt + C$$

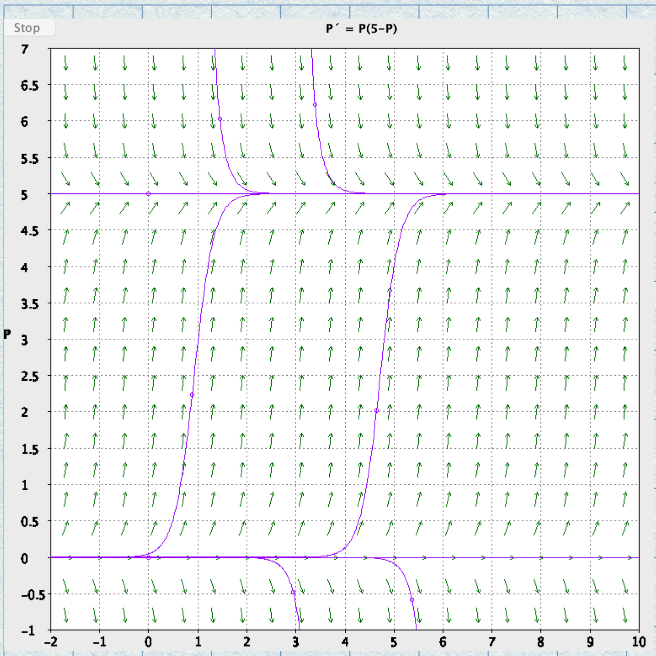
$$\Rightarrow \frac{1}{M} \ln P - \frac{1}{M} \ln(M-P) = kt + C$$

$$\Rightarrow \frac{1}{M} \ln\left(\frac{P}{M-P}\right) = kt + C$$

Solve for P:

$$\boxed{P(t) = \frac{M P_0}{P_0 + (M - P_0)e^{-Mkt}}} \quad \text{(check)}$$

$P_0 \rightarrow$ population at time $t = 0$



Solution to 2nd Example.

We were left at $\frac{dy}{dx} = y^2 + C_1$. ← separable.

$$\Rightarrow \int \frac{dy}{y^2 + C_1} = \int dx$$

Integral on LHS depends on the sign of C_1!

Case 1: $C_1 = 0$

$$\int \frac{dy}{y^2} = \int dx \Rightarrow -\frac{1}{y} = x + C_2$$

$$\Rightarrow y = -\frac{1}{x + C_2}$$

Case 2: $C_1 > 0$

$$\int \frac{dy}{y^2 + C_1} = \int dx$$

$$\Rightarrow \quad \frac{1}{c_1}\int \frac{dy}{\left(\frac{y}{\sqrt{c_1}}\right)^2+1} = \int dx$$

$$\Rightarrow \quad \frac{1}{\sqrt{c_1}} \arctan\left(\frac{y}{\sqrt{c_1}}\right) = x + C_2$$

$$\Rightarrow \quad y = \sqrt{c_1} \tan\left(\sqrt{c_1}(x+C_2)\right)$$

Case 3: $c_1 < 0$

$$\int \frac{dy}{y^2-|c_1|} = \int dx$$

$$\int \frac{1}{\sqrt{|c_1|}}\left(-\frac{1}{y+\sqrt{|c_1|}} + \frac{1}{y-\sqrt{|c_1|}}\right)dy = \frac{1}{2}x^2 + C_2$$

$$\frac{1}{\sqrt{|c_1|}} \ln\left(\frac{y-\sqrt{|c_1|}}{y+\sqrt{|c_1|}}\right) = \frac{1}{2}x^2 + C_2$$