Plan for today:
1. Finish 2.1
2. Start 2.2

Learning Goals
1. Be able to set up and solve a differential equation which describes the growth of a population based on information given by the problem in hand. Such a differential equation may or may not be a logistic equation.
2. Be able to recognize stable, unstable and semistable critical points of an autonomous ODE
3. Be able to identify the critical points and equilibrium solutions of an autonomous differential equation and construct a phase diagram.
4. Be able to extract qualitative information about the behavior of solution of an autonomous differential equation from it phase diagram
5. Be able to construct a bifurcation diagram for a differential equation depending on a parameter

Reminders-Announcements
1. HW 09 moved to Thursday. HW 07 and HW 08 still due Tuesday.
2. No class on Wednesday (Reading Day)
3. Read the textbook!

Saw: Logistic eq'n: $a, b > 0$

$$\frac{dP}{dt} = aP - bP^2 = kP(M-P)$$

$$\left(k = b,\ M = \frac{a}{b}\right)$$

Assumption: Birth rate decreases linearly as population increases; death rate const.

Found: $$\boxed{P(t) = \frac{M P_0}{P_0 + (M-P_0)e^{-Mkt}}}$$

$P_0 \to$ population at time $t=0$

Observed: $P(t) \equiv 0$, $P(t) \equiv M$ are always sol's.

Other situations modeled by logistic eq'n:

Competition: death rate increases linearly as pop. increases.

Zombies: $P_0 \to$ pop. of city

$N(t) \to$ # of zombies

$$\frac{dN}{dt} = k \underbrace{N(t)}_{\#\text{ zombies}} \underbrace{(P_0 - N(t))}_{\text{non-zombies}}$$

<u>Ex:</u> Population of Jackalopes

$$\frac{dP}{dt} = aP - bP^2$$

Initially: 120

At $t=0$

births/month $= aP = 8$

deaths/month $= bP^2 = 6$

Find a, b to describe eq'n.

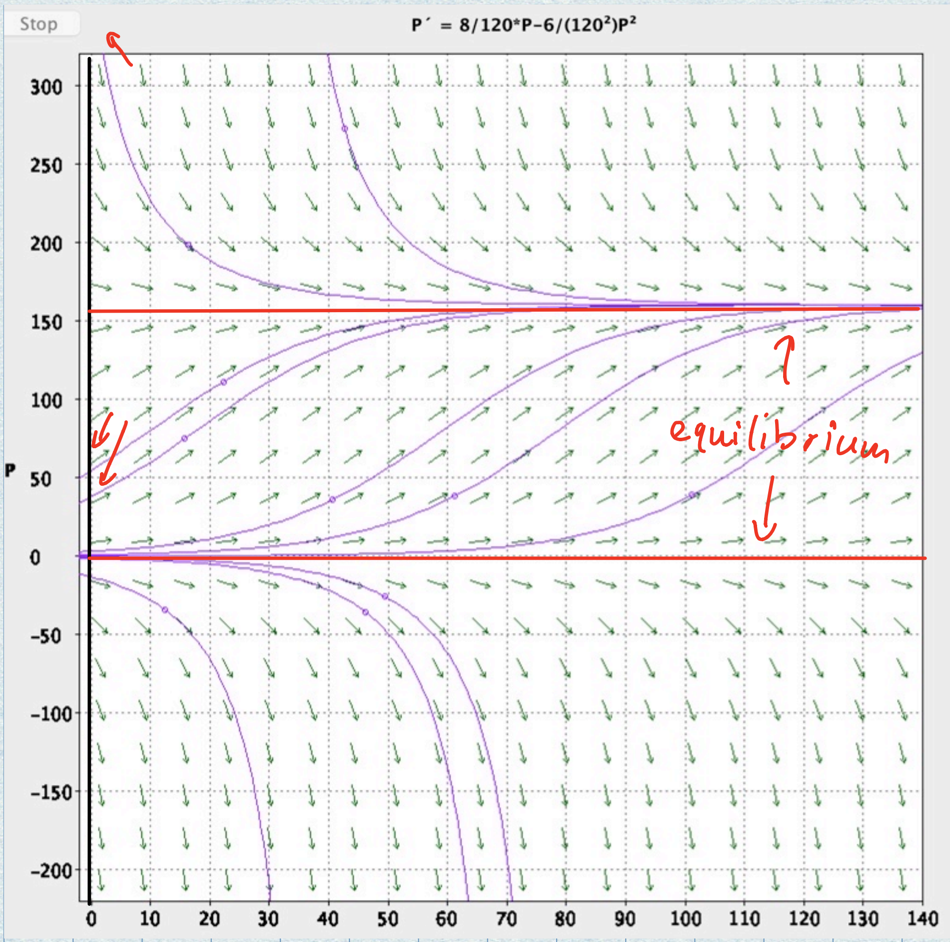
$$\left(\frac{dP}{dt} = (\underbrace{\beta}_{\text{births/(unit pop)(unit of time)}} - \delta) \underbrace{P}_{\text{pop.}} \right)$$

$$aP = 8 \Rightarrow a = \frac{8}{120}$$

$$bP^2 = 6 \Rightarrow b = \frac{6}{120^2}$$

So: $$\frac{dP}{dt} = \frac{8}{120}P - \frac{6}{120^2}P^2 = \frac{6}{120}P\left(\frac{\frac{8}{120}}{\frac{6}{120^2}} - P\right)$$

$$= \frac{6}{120}P(160 - P)$$



160

limiting population/ carrying capacity

If $P_0 > 0$ then $P(t) \longrightarrow 160$

If $P_0 = 0$ then $P(t) \equiv 0$

If $P_0 < 0$ (nonphysical), sol'n has a vertical asymptote.

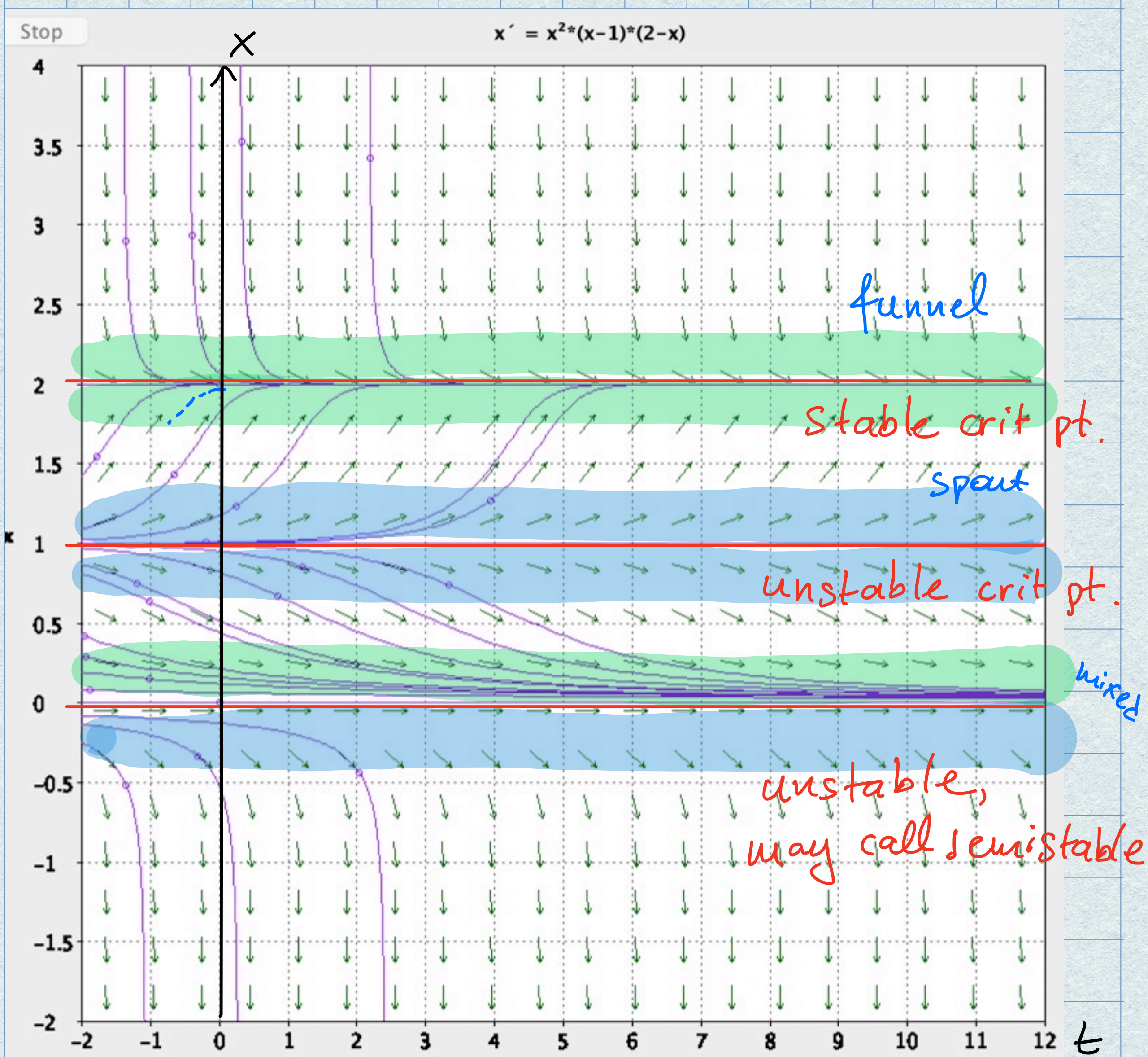
<u>Def'n:</u> Autonomous dif eq'n: $\frac{dy}{dt} = \underbrace{f(y)}_{\text{no } t}$

The values c such that $f(c) = 0$ are called critical points.

(Above: $\frac{dP}{dt} = \frac{6}{120} P(160 - P)$: crit $\rightarrow 0, 160$)

$y(t) = c$, c critical, is an <u>equilibrium solution</u>.

<u>Ex:</u> $x'(t) = x^2(x-1)(2-x)$ <u>Autonomous</u>



Critical pts: 0, 1, 2

"Def'n" $\frac{dx}{dt} = f(x)$ (*) autonomous dif. eq'n.

$x = c \rightarrow$ critical pt, called

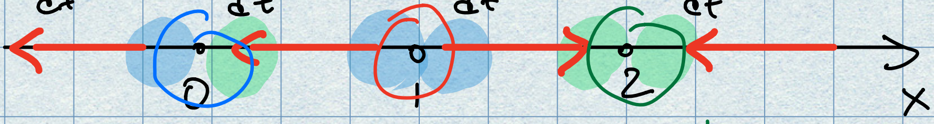
stable: if x_0 is close enough to c then the sol'n $x(t)$ of (*) w/ $x(0) = x_0$ stays close to c for all $t \geqslant 0$.

unstable: otherwise.

<u>Phase diagrams</u>: helps understand stable/unstable crit. pts.

<u>Ex:</u> $\frac{dx}{dt} = x^2(x-1)(2-x)$ <u>Autonomous</u>

$\frac{dx}{dt} < 0$ $\frac{dx}{dt} < 0$ $\frac{dx}{dt} > 0$ $\frac{dx}{dt} < 0$ $\rightarrow$ der. of a sol'n which starts at $x_0 \in (2, \infty)$ is negative



unstable/semistable one towards & one away

unstable (arrows point away)

stable (arrows point towards)

Stable

Unstable

Semistable