Plan for today:
Finish 2.2
Start 2.3

Learning goals for the day:

1. Be able to construct a bifurcation diagram for a differential equation depending on a parameter
2. Be able to set up as solve a differential equation for the motion of a body under the influence of air resistance.

Reminders-Announcements

1. Ungraded Quiz 1.5 on Monday.
2. Quiz 2 on Thursday
3. Future quizzes are open book, see syllabus update
4. Read the textbook!

Last time: critical pts of an autonomous eqn.

$$\frac{dx}{dt} = f(x)$$

c critical: $f(c) = 0$

Logistic Model w/ harvesting

fish in a lake following a logistic model, harvest h every year.

$$\frac{dx}{dt} = \underbrace{x(100-x)}_{\text{logistic part}} - h$$

$x(t) \to$ # of fish in year t

autonomous

depends on parameter

Crit pts? When is $x(100-x) - h = 0$?

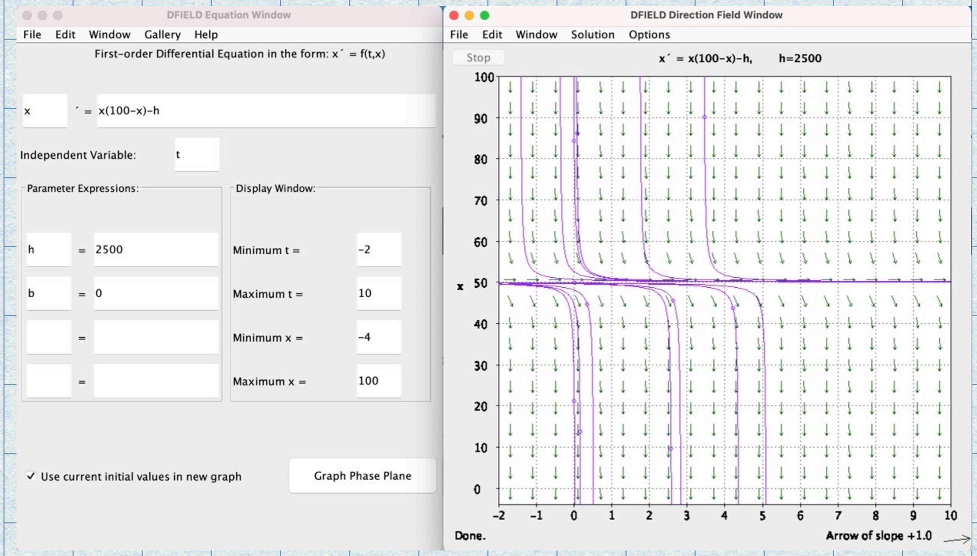
$$-x^2 + 100x - h = 0$$

$$\Rightarrow \quad x = \frac{100 \pm \sqrt{10{,}000 - 4h}}{2} \quad = \Delta$$

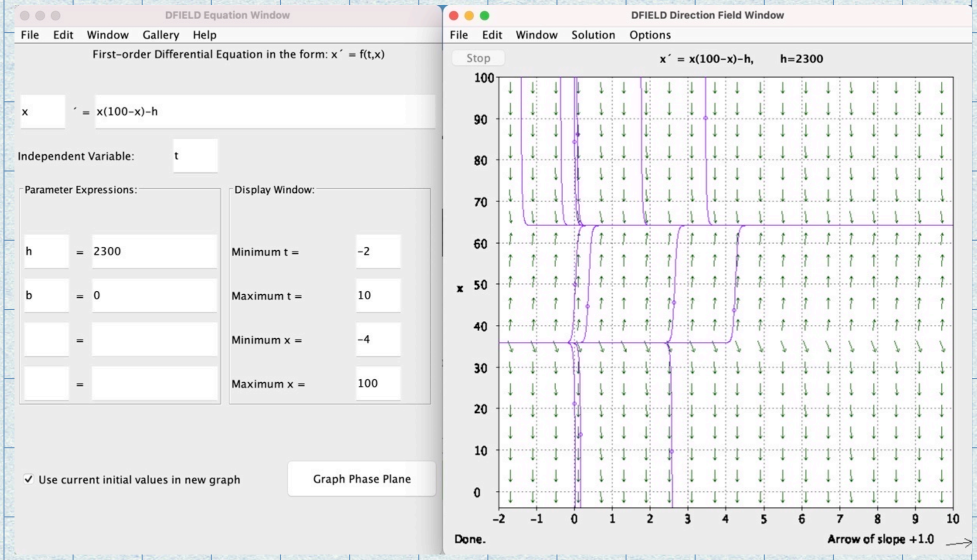
(*)

# of critical pts depends on h.

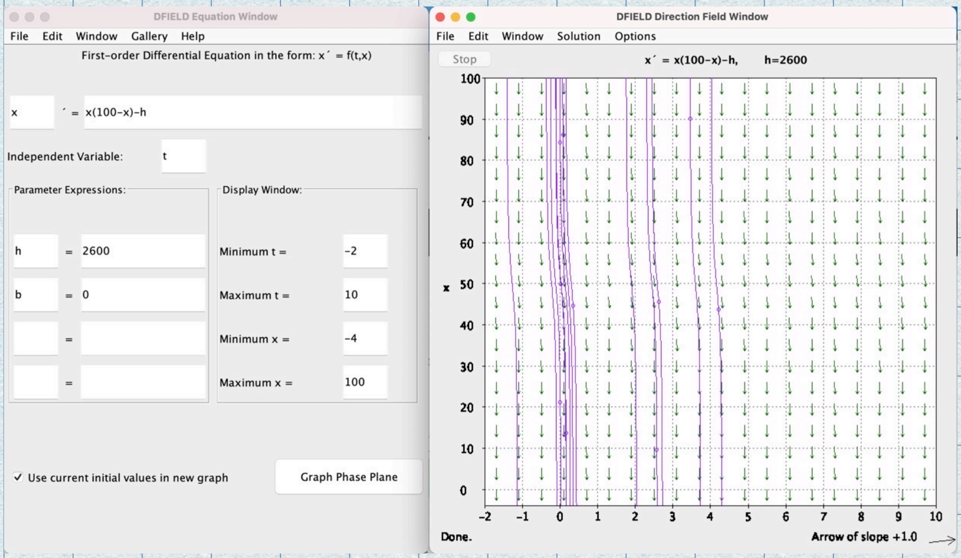
If $h = 2500$ then $10000 - 4h = 0$ and we have only 1 crit. pt.



If $h < 2500$ then $\Delta > 0$ and $\rightarrow$ 2 crit pts



If $h > 2500$ then $\Delta < 0$ and we have no crit pts.

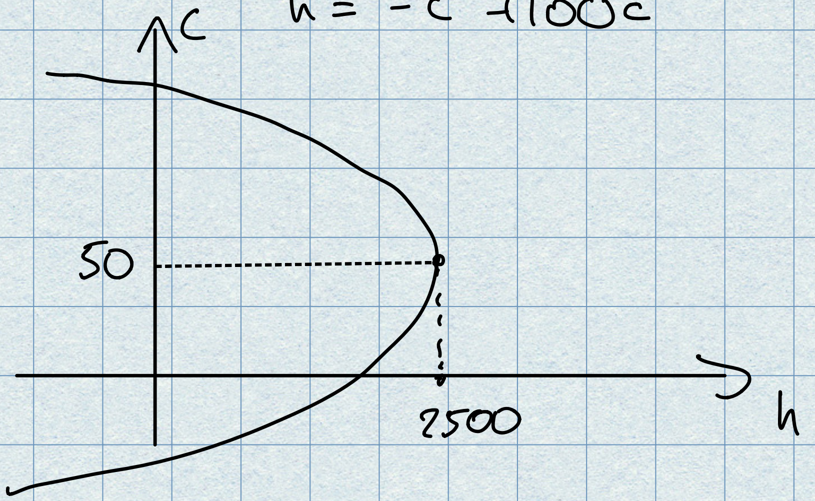


$h = 2500 \rightarrow$ bifurcation pt: qualitative behavior of eqn changes between $h < 2500$ & $h > 2500$

Bifurcation Diagram: sets of pts (h, c) so that c is a crit. pt.

(x) $c^2 - 100c + h = 0$

$h = -c^2 + 100c$



2.3 Acceleration & velocity models.

In general: air resistance

$$F_R = k\, v^p \qquad 1 \le p \le 2$$

F_R ↑ resistance, k ↑ a const., v^p ← velocity

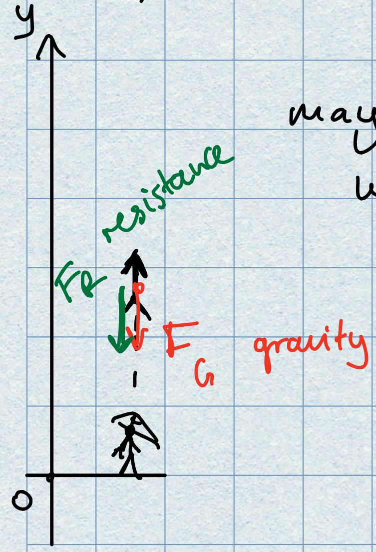
F_R always in direction opposite to the motion

$k \rightarrow$ depends on viscosity / density of air & shape of body.

Ex: Arrow shot straight upwards, initial velocity $v_0 = 160\ ft/s$.

deceleration due to air resistance: $\frac{v^2}{800} \frac{ft}{s^2}$

a) how high does it go?



maybe $\frac{d^2y}{dt^2} = \ldots$; easier to work w/ velocity directly.

$$\underset{\substack{\uparrow \\ \text{acceleration}}}{\frac{dv}{dt}} = \underbrace{-\frac{v^2}{800}}_{\text{air resistance}} - \underbrace{32}_{\text{gravitational acceleration}}$$

autonomous, seperable!

Exerccie: solve! (soln at the end)

find: $v(t) = 160 \tan\left(-\frac{t}{5} + c\right)$

Know: $v(0) = 160 \rightarrow c = \frac{\pi}{4}$

$160 = 160 \tan(c)$

$\tan(c) = 1$

How do we find when it will be at its highest point?

Want $v(t) = 0 \iff 160\tan\left(-\frac{t}{5} + \frac{\pi}{4}\right) = 0$

$\Rightarrow t = \frac{5\pi}{4}$

$$y(t) = \int v(t)\,dt = \int 160\tan\left(-\frac{t}{5} + \frac{\pi}{4}\right) dt$$

$$y(t) = 800\ln\left(\cos\left(\frac{\pi}{4} - \frac{t}{5}\right)\right) + C_2$$

Find C_2: $y(0) = 0$

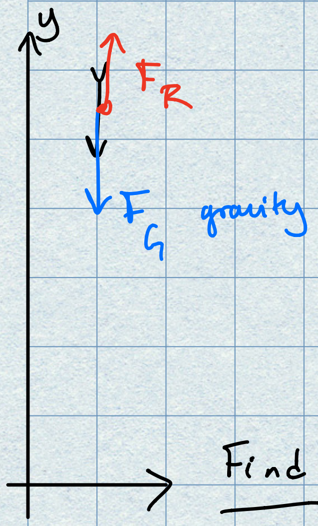
$\Rightarrow \ldots \Rightarrow C_2 = 400\ln(2)$

So: highest we reach

$$y\left(\frac{5\pi}{4}\right) = \underbrace{800\ln\Big(\underbrace{\cos\big(\overbrace{\tfrac{\pi}{4} - \tfrac{\pi}{4}}^{0}\big)}_{1}\Big)}_{0} + 400\ln(2)$$

$$= 400\ln(2).$$

b) When arrow is going down.



$$\frac{dv}{dt} = \underbrace{-32 + \frac{v^2}{800}}, \quad v \text{ negative} \qquad d_R = \frac{v^2}{800}$$

Solve (exercise)

Find: $v(t) = -160\,\dfrac{1 - e^{-2t/5}}{1 + e^{-2t/5}} \xrightarrow{t \to \infty} -160$

Reach limiting velocity.

time measured from the moment when arrow is at highest altitude

Sol's of examples

1. $$\frac{dv}{dt} = -32 - \frac{v^2}{800}$$

$$\frac{dv}{dt} = -32\left(1 + \frac{v^2}{25600}\right)$$

$$= -32\left(1 + \left(\frac{v}{160}\right)^2\right)$$

$32 \cdot 800 = 25600 = 160^2$

Separate variables:

$$\int \frac{dv}{1+\left(\frac{v}{160}\right)^2} = -\int 32\,dt, \text{ set } \frac{v}{160} = z$$

$$\int \frac{160\,dz}{1+z^2} = -\int 32\,dt$$

$$\Rightarrow \arctan(z) = -\frac{1}{5}t + C$$

$z = 160v$

$$\Rightarrow v = 160\tan\left(-\frac{t}{5} + C\right)$$

2. $$\frac{dv}{dt} = -32 + \frac{v^2}{800} = -32\left(1 - \frac{v^2}{25{,}600}\right)$$

$$\Rightarrow \int \frac{dv}{1-\left(\frac{v}{160}\right)^2} = \int -32\,dt \Rightarrow$$

$z = \frac{v}{160}$

$$\Rightarrow \int \frac{160\,dz}{1-z^2} = -32t + C$$

$$\Rightarrow 160 \int \frac{1}{2}\frac{1}{1-z} + \frac{1}{2}\frac{1}{1+z}\,dz = -32t + C$$

$$\Rightarrow \quad 160\left(-\frac{1}{2}\ln|1-z| + \frac{1}{2}\ln|1+z|\right) = -32t + C$$

$$\Rightarrow \quad 80 \ln\left|\frac{1+z}{1-z}\right| = -32t + C$$

$$\Rightarrow \quad \ln\left|\frac{1+z}{1-z}\right| = -\frac{2}{5}t + C_1 \qquad \left(C_1 = \frac{C}{80}\right)$$

$$\Rightarrow \quad \frac{1+z}{1-z} = \pm e^{C_1} e^{-\frac{2}{5}t}$$

$$\Rightarrow \quad \frac{1+z}{1-z} = C_2 e^{-\frac{2}{5}t} \qquad \left(C_2 = \pm e^{C_1}\right)$$

$$\Rightarrow \quad z\left(C_2 e^{-\frac{2}{5}t} + 1\right) = C_2 e^{-\frac{2}{5}t} - 1$$

$$\Rightarrow \quad v = 160 \, \frac{C_2 e^{-\frac{2}{5}t} - 1}{C_2 e^{-\frac{2}{5}t} + 1}$$

Starting to measure time from the moment when the arrow reaches its highest altitude, $v(0) = 0$.

So $\quad C_2 - 1 = 0 \Rightarrow C_2 = 1$

$$v = 160 \, \frac{e^{-\frac{2}{5}t} - 1}{e^{-\frac{2}{5}t} + 1}$$