Plan for Today

Discuss 2.4-2.5

Learning Goals:

1. Be able to carry out a few steps of the Euler method and the improved Euler method by hand.
2. Be able to carry out more using a computer algebra system.
3. How can we obtain a more accurate approximation when using Euler ' s method?

Reminders-Announcements

1. Read the textbook!
2. Quiz 1.5 (ungraded) available at 9.30 am today
3. Quiz 2 (graded) this Thursday
4. Review on Wednesday. Go over the worksheet and prepare questions.

§2.4-2.5 Euler's Method, Improved Euler's Method

Algorithms for solving 1st order ODEs approximately.

Why we need them: can't solve most ODEs explicitly.

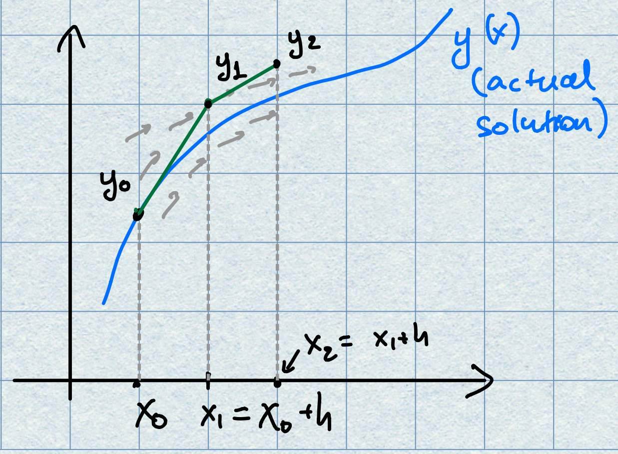
Ex: $\frac{dx}{dt} = e^{-t^2} \Rightarrow x(t) = \int e^{-t^2} dt + C$ can't write "nicely"

Euler's Method

Trying to approximate: sols of $\begin{cases} \frac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \end{cases}$

on an interval.

Idea: $y(x+h) \sim y(x) + h \left.\frac{dy}{dx}\right|_x = y(x) + h\underbrace{f(x,y)}_{\text{from ODE.}}$



Set $x_1 = x_0 + h$, h a fixed small step size.

Set

$$y_1 = y_0 + hf(x_0, y_0)$$

__Next:__ set $x_2 = x_1 + h$

$$y_2 = y_1 + h f(x_1, y_1)$$

More generally:

fix step size h.

Set $x_{n+1} = x_n + h$

$$y_{n+1} = y_n + h f(x_n, y_n) \longleftarrow$$ y_j approximate true values $y(x_j)$

__Ex:__ $\begin{cases} \dfrac{dy}{dx} = x + y \\ y(0) = 1 \end{cases}$

Take $h = 0.5$; $x_1 = x_0 + h = 0 + 0.5 = 0.5$

$x_0 = 0$; $y_1 = y_0 + h\,(x+y)\big|_{(x_0, y_0)}$

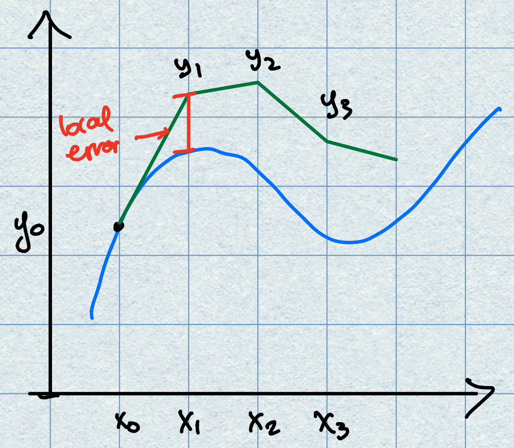
$y_0 = 1$; $= 1 + 0.5\,(1 + 0)$

$= 1.5$

$y_2 = y_1 + h\,(x_1 + y_1)$

$= 1.5 + 0.5\,(1.5 + 0.5)$

__Note:__ There is an error in the approximation.

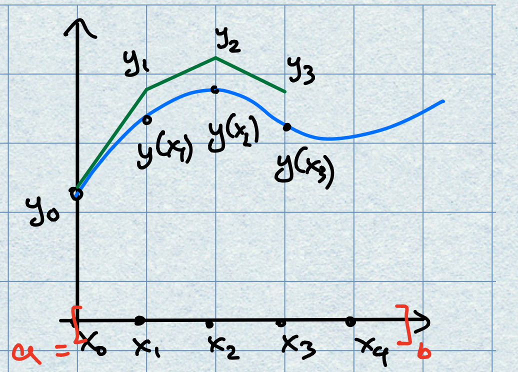


Error introduced at every step!

local errors build up, "cumulative error"

Goal: control the errors.

Main way: shrink step size.



Thm: Given IVP

$$\begin{cases} \dfrac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \end{cases} \quad (*)$$

Suppose: 1. $(*)$ has unique sol'n $y(x)$ on $[a,b]$, $a = x_0$

2. $\dfrac{d^2y}{dx^2}$ cont. on $[a,b]$

Then there is a constant C → doesn't depend on h so that:

$x_1, \ldots, x_n \in [a,b]$ $\quad (x_j = x_0 + jh)$

$y_1, \ldots, y_n$ approx. of $y(x_1), \ldots, y(x_n)$ coming from Euler's method

Then $\quad |y_k - y(x_k)| \leq Ch \quad k = 0, 1, \ldots, n$

So: if assumptions of the theorem are satisfied, we can choose h small enough to make the error as small as we please.

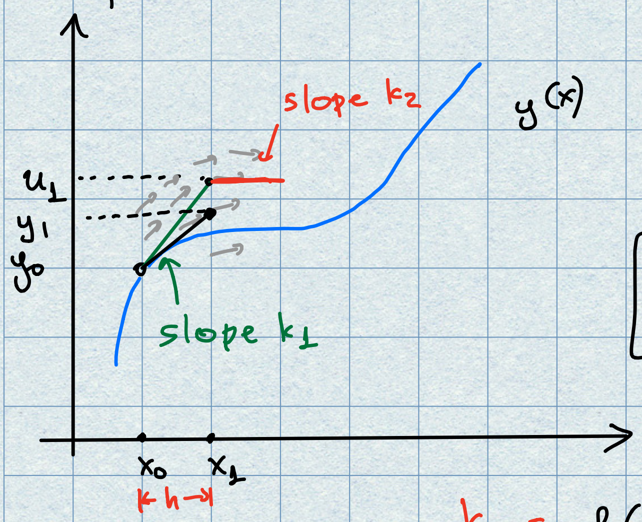
Sup. want error < 0.1

thm says there is const which depends on ODE might be large say $C = 1000$

If we make $h < \dfrac{1}{10000}$ then $|y_k - y(x_k)| < \dfrac{1}{10}$

Mention: smaller step size means more iterations → more computational time.

Improved Euler Method



$$\begin{cases} \dfrac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

fix step size h.

$$\begin{cases} x_1 = x_0 + h \\ u_1 = y_0 + h\underbrace{f(x_0, y_0)}_{k_1} \leftarrow \text{predictor} \end{cases}$$

$$= y_0 + h\,k_1$$

$k_2 = f(x_1, u_1) \rightarrow$ slope of the graph of the sol'n passing through (x_1, u_1)

Go back: set

$$y_1 = y_0 + h\,\underbrace{\frac{k_1 + k_2}{2}}_{} \leftarrow \text{corrector}$$

average of slope at (x_0, y_0) & slope predicted by Euler's method at (x_1, u_1)

<u>More generally:</u> $h \rightarrow$ step size

$$x_{n+1} = x_n + h$$

$$u_{n+1} = y_n + h\,f(x_n, y_n) \quad \begin{pmatrix}\text{as in} \\ \text{Euler}\end{pmatrix}$$

$$y_{n+1} = y_n + h\,\frac{f(x_n, y_n) + f(x_{n+1}, u_{n+1})}{2}$$

Error: $|y(x_n) - y_n| \leq C h^2$ (under assumptions)

If $C = 1000$, want error < 0.1, suffices to take $h = \frac{1}{100}$