Plan for Today:
Start 3.1

Learning Goals:
1. Be able to recognize a 2nd order linear ODE.
2. Know the terminology: homogeneous/nonhomogeneous second order ODE (!! Unrelated to homogeneous 1st order equations in § 1.6!)
3. What is the superposition principle?
4. What does the theorem of existence and uniqueness for IVPs for 2nd order linear ODEs say?
5. What do solutions to linear homogeneous 2nd order ODEs look like?

Reminders/announcements:
Quiz 2 closes in 1 hour
HW 13 will be extended

In the next few lessons, writing in green color will contain optional connections to linear algebra

Ch 3

3.1 2nd order ODE.

Until now: reducible 2nd order ODE

$G_1(y'', y', y) = 0$ Ex: $y'' + yy' = 0$

$G_2(y'', y', x) = 0$ Ex: $xy'' + 3y' = 5$

Today: linear 2nd order ODE

$$A(x)y'' + B(x)y' + C(x)y = F(x) \quad (*)$$

Ex:

$$e^x y'' + \cos(x) y' + 3y = 3\ln(x) \quad (*)$$

Non ex: $y'' + 3y^2 = 0$

Terminology: if $F(x) = 0$ in $(*)$ then $(*)$ is called a homogeneous 2nd order linear ODE [unrelated to homogeneous $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$]

If $F(x) \neq 0$ then (*) called non-homogeneous / inhomogeneous

(*) non-homogeneous.

$$e^x y'' + \cos(x) y' + 3y = 0$$

homog. (the homog. eq'n associated to (*))

From now on: divide by $A(x)$

$$y'' + \underset{\substack{\parallel \\ B/A}}{p(x)} y' + \underset{\substack{\parallel \\ C/A}}{q(x)} y = \underset{\substack{\parallel \\ F/A}}{f(x)}$$

Seen: when solving reducible 2nd order eq's there were 2 free parameters.

$$y'' = 0 \Rightarrow y' = c_1 \Rightarrow y = c_1 x + c_2$$

Expect: 2 pieces of info needed to specify a sol'n.

What 2 pieces of info are good? (what should an appropriate IVP look like to have existence & uniqueness?)

"Bad" info:

$$\begin{cases} y'' + y = 0 \\ y(0) = 0, \quad y(2\pi) = 0 \end{cases}$$

check: $y = A\sin(x)$ solves IVP for any A.

$$y'' = -A\sin(x) = -y$$

Prescribing values at different locations might not give a unique sol'n!

The "good" info

Given $y'' + p(x)y' + q(x)y = f(x)$ (**)

p, q, f cont. on interval I.

$a \in I$, $b_1, b_2 \in \mathbb{R}$

then there is a unique sol'n to (**)

satisfying $\begin{cases} y(a) = b_1 \\ y'(a) = b_2 \end{cases}$ (***)

and it is defined on all of I

(compare w/ Ex. & Un. thm in 1.5)

(**) + (***) called an IVP for 2nd order linear eq's.

Recall: 1st order linear there was a formula for sol's. (integrating factor etc)

what do sol's look like?

1. Superposition principle.

Linear Homog. eq'n: $y'' + p(x)y' + q(x)y = 0$

Ex: $y'' + y = 0$

If y_1, y_2 are sols to homog. eq'n, then $c_1 y_1 + c_2 y_2$ is also a sol'n.

c_1, c_2 are constants!

linear combination

In ex: $y_1 = \sin(x)$

$y_2 = \cos(x)$ $\qquad y_2'' = \cos'' = -\cos = -y_2$

$$(c_1 y_1 + c_2 y_2)'' = (c_1 \sin(x) + c_2 \cos(x))''$$
$$= c_1 (\sin(x))'' + c_2 (\cos(x))''$$
$$= -c_1 \sin(x) - c_2 \cos(x)$$
$$= -(c_1 y_1 + c_2 y_2)$$

so $c_1 y_1 + c_2 y_2$ is a sol'n

[Superposition pr. says that the solutions of linear homog. 2nd order ODE form a vector space]

Sup. pr: if we have some sols we can produce more.

Ex: $\begin{cases} y'' + y = 0 \\ y\left(\frac{\pi}{4}\right) = 1 \\ y'\left(\frac{\pi}{4}\right) = 2 \end{cases}$ $\rightarrow$ unique sol'n exists bec. of theorem

Try to create this sol'n as a linear comb. of $y_1 = \sin(x)$, $y_2 = \cos(x)$.

$$y = c_1 \sin(x) + c_2 \cos(x)$$

What are the c_1, c_2 for which satisfies IVP, if any?

$$y\left(\frac{\pi}{4}\right) = c_1 \frac{\sqrt{2}}{2} + c_2 \frac{\sqrt{2}}{2} = 1 \qquad (1)$$

$$y' = c_1 \cos(x) - c_2 \sin(x)$$

$$\Rightarrow y'\left(\frac{\pi}{4}\right) = c_1 \frac{\sqrt{2}}{2} - c_2 \frac{\sqrt{2}}{2} = 2 \qquad (2)$$

(1), (2) $\Rightarrow$

$$2c_1 \frac{\sqrt{2}}{2} = 3 \Rightarrow c_1 = \frac{3}{\sqrt{2}}$$

$$2c_2 \frac{\sqrt{2}}{2} = -1 \Rightarrow c_2 = -\frac{1}{\sqrt{2}}$$

So: $y = \frac{3}{\sqrt{2}} \sin(x) - \frac{1}{\sqrt{2}} \cos(x)$ is my sol'n.!

<u>Questions:</u> Can we produce <u>any</u> sol'n to an IVP from linear combinations of a pair of sol's?
Are all pairs good enough?