Plan for Today:
3.1

Learning goals:
Concepts: Linearly Independent Solutions, Wronskian, Characteristic Equation

Skills:
1. Be able to find the solution to an IVP for a linear 2nd order ODE given two linearly independent solutions for the ODE
2. Be able to check if two solutions are linearly independent using the Wronskian

Reminders-Announcements
1. Read the textbook!
2. Office hours today and tomorrow
3. Quiz 2 grades posted.

Last time: $y'' + p(x)y' + q(x)y = 0$

Saw

$$\begin{cases} y'' + y = 0 \\ y\left(\frac{\pi}{4}\right) = 1 \\ y'\left(\frac{\pi}{4}\right) = 2 \end{cases} \qquad (1)$$

can be solved by setting $y = c_1 \sin(x) + c_2 \cos(x)$ and finding c_1, c_2.

what makes them good building blocks?

Q: Can we use any two solutions of a linear 2nd order eqn as building blocks to produce any solution?

Ex: $y_1 = \sin(x)$, $\tilde{y}_2 = 3\sin(x)$

Both solve $y'' + y = 0$

Can we solve (1) as a linear comb. of $y_1, \tilde{y}_2$?

$$c_1 \, y_1\left(\frac{\pi}{4}\right) + c_2 \, \tilde{y}_2\left(\frac{\pi}{4}\right) = 1$$

$$c_1 \, y_1'\left(\frac{\pi}{4}\right) + c_2 \, \tilde{y}_2'\left(\frac{\pi}{4}\right) = 2$$

$$\left.\begin{aligned} c_1 \frac{\sqrt{2}}{2} + c_2 \cdot 3\frac{\sqrt{2}}{2} &= 1 \\ c_1 \frac{\sqrt{2}}{2} + c_2 \, 3\frac{\sqrt{2}}{2} &= 2 \end{aligned}\right\} \Rightarrow \text{can't find such } c_1, c_2.$$

Def'n: 2 functions f_1, f_2 on an interval I are called linearly independent if:

$$\underbrace{c_1 f_1(x) + c_2 f_2(x)}_{\text{linear combination}} = 0 \text{ on } I \implies c_1 = c_2 = 0$$

(where c_1, c_2 constants)

Equivalent: None of them is a const. multiple of the other.

If not linearly indep. $\longrightarrow$ linearly dependent.

Check: $\sin(x), \cos(x)$ lin. indep. bec. none is const. mult. of the other.

$\sin(x), 3\sin(x)$ lin. dependent.

Thm: Let y_1, y_2 be two linearly independent sols of the homogeneous

$$y'' + p(x)\,y' + q(x)\,y = 0 \qquad 2$$

$p(x), q(x)$ cont. on an interval I.
If y is any sol'n to 2 then there are c_1, c_2 such that

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

"Any 2 lin. indep. sols are good building blocks"

[Sols of 2 form a 2-dim'l vector space, any 2 lin. indep. sols are a basis]

Ex: $x^2 y'' - xy' + y = 0$ 3

$$\left.\begin{aligned} y_1 &= x \\ y_2 &= x \ln x \end{aligned}\right\} \text{ sols on } I = (0, \infty).$$

y_1, y_2 lin. independent (none of them const. multiple of the other)

Find sol'n of 3 w/ $y(1) = 2$, $y'(1) = 3$.

write $y = c_1 y_1 + c_2 y_2$, find c_1, c_2.

$$y(1) = c_1 \underbrace{1}_{y_1(1)} + c_2 \cdot \underbrace{0}_{y_2(1)} = 2$$

$$y'(1) = c_1 \cdot \underbrace{1}_{y_1'(1)} + c_2 \cdot 1 = 3$$

$y_2'(x) = \ln x + 1$

<u>Find:</u> $c_2 = 1$

$c_1 = 2$

<u>A way to check linear independence:</u>
Wronskian determinant.

<u>Sup.</u> y_1, y_2 sols of $y'' + p(x)y' + q(x)y = 0$

$$y = c_1 y_1(x) + c_2 y_2(x)$$

when does y solve the IVP $y(a) = b_1$, $y'(a) = b_2$

Try to find c_1, c_2.

$$y(a) = b_1 \Rightarrow c_1 \, y_1(a) + c_2 \, y_2(a) = b_1$$
$$y'(a) = b_2 \Rightarrow c_1 \, y_1'(a) + c_2 \, y_2'(a) = b_2$$

known

unknown

$$\begin{pmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Can find a unique pair of c_1, c_2 exactly when

$$\underbrace{\det \begin{pmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{pmatrix}}_{\text{Wronskian determinant.}} \neq 0.$$

$$W(y_1, y_2) = \det \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}$$
$$= y_1 y_2' - y_2 y_1'$$

Thm: If y_1, y_2 sols to
$$y'' + p(x)y' + q(x)y = 0$$
on I, p, q cont.

→ y_1, y_2 lin. dep. on I ⟹ $W(y_1,y_2) \equiv 0$ everywhere on I.

→ y_1, y_2 lin indep. on I ⟹ $W(y_1,y_2) \neq 0$ everywhere on I.

Ex:
$$W(\cos(x), \sin(x)) = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$
sols of $y'' + y = 0$

$$= \cos^2(x) + \sin^2(x) = 1$$
never 0.

⟹ $\cos(x), \sin(x)$ lin. indep.

$$W(x, x\ln x) = \begin{vmatrix} x & x\ln x \\ 1 & \ln x + 1 \end{vmatrix}$$
sols

$x^2 y'' - xy' + y = 0$

$$= x\ln x + x - x\ln x$$
$$= x \neq 0 \text{ on } (0, \infty)$$

! The W of two lin. independent functions (not solutions of the same eq'n) can vanish only at one pt.
$$W(x, \sin(x)) = \begin{vmatrix} x & \sin(x) \\ 1 & \cos(x) \end{vmatrix}$$

$= x\cos(x) - \sin(x)$

$W(x, \sin(x))\big|(0) = 0$, but $x, \sin(x)$ lin. indep. on $(-\infty, \infty)$,

but they are <u>not</u> sols of the same 2nd order linear eq'n (no contradiction).

Rmk: If $y_1 = c\, y_2$ (so y_1, y_2 not lin. indep)

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & c y_2(x) \\ y_1'(x) & c y_2'(x) \end{vmatrix} =$$

$$= c y_1(x) y_2'(x) - c y_1'(x) y_2(x)$$
$$= 0.$$

(how to see that lin. dependence $\Rightarrow$ Wronskian 0.)