Plan for today:

Finish 3.1

Start 3.2

Learning goals:

Concepts: Characteristic Equation, linearly independent functions, existence and uniqueness, Method of reduction

3.2, HW

Skills:

1. Be able to solve a homogeneous constant coefficient linear 2nd order ODE whose characteristic equation has two distinct real roots or one repeated root.
2. Be able to use reduction of order to find a second linearly independent solution of a given 2nd order linear equation when one solution is known.
3. Given three linearly dependent functions, be able to write one as a linear combination of the rest

Reminders:

1. Read the textbook!

Const. Coef. 2nd order linear homog.

$$ay'' + by' + cy = 0 \qquad (1)$$

const.

Consider $y = e^{rx}$, r const.

$$(1) \Rightarrow ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0$$

$$(ar^2 + br + c)e^{rx} = 0$$

so if we can find r so that

$$ar^2 + br + c = 0 \quad \leftarrow \text{Characteristic Eq'n}$$

then e^{rx} will be a sol'n to (1).

1st case: Char. eq'n has 2 real distinct roots.

then $e^{r_1 x}, e^{r_2 x}$ where r_1, r_2 are the roots are 2 lin. indep. sols.

Ex: $2y'' + 3y' = 0$ (2)

$$2r^2 + 3r = 0$$

$$(2r+3)r = 0 \Rightarrow r = 0 \text{ or } r = -\frac{3}{2}$$

So: $y_1 = e^{0x} = 1$, $y_2 = e^{-\frac{3}{2}x}$ are lin. indep. sols of (2). Any sol'n of (2) is written as

$$y = c_1 + c_2 e^{-\frac{3}{2}x}$$

2nd case: One repeated real root.

Ex: $y'' - 4y' + 4y = 0$ (3)

$$r^2 - 4r + 4 = 0 \rightarrow (r-2)^2 = 0$$

$r = 2$, repeated.

$y = e^{2x}$ is one sol'n.

How do we find a second lin. indep. sol'n?

Interlude: Method of reduction. Appears in HW 3.2

Given Linear eq'n such as

$$y'' - 4y' + 4y = 0 \quad (4)$$

(coef. may not be const. for method of red.)

and one sol'n is known.

(in our case $y_1 = e^{2x}$) we can find a second lin. indep. sol'n by setting $\underline{y = v(x)\, y_1(x)}$ and finding v.

↳ plug into (4)

$$\left.\begin{aligned} y' &= v'y_1 + vy_1' \\ y'' &= v''y_1 + 2v'y_1' + vy_1'' \end{aligned}\right\}$$

(4)

$$\Rightarrow \left(v''y_1 + 2v'y_1' + vy_1''\right) - 4\left(v'y_1 + vy_1'\right) + 4vy_1 = 0$$

y_1 sol'n.

$$v''y_1 + 2v'y_1' - 4v'y_1 + v\cancel{\left(y_1'' - 4y_1' + 4y_1\right)} = 0$$

$$v''e^{2x} + \cancel{2v' \cdot 2e^{2x}} - \cancel{4v'e^{2x}} = 0$$

$$\Rightarrow v'' = 0 \quad \Rightarrow v = ax + b$$

So xe^{2x}, e^{2x} are 2 lin. indep. sols.

$(ax+b)e^{2x}$, e^{2x} lin. indep. if $a \neq 0$ //

<u>So</u>: if char. eq'n has a repeated root r_0, then $e^{r_0 x}$, $xe^{r_0 x}$ are two lin. indep. sols. <u>Any</u> sol'n is of form

$$y = c_1 e^{r_0 x} + c_2 x e^{r_0 x}.$$

(*)

<u>For Exams:</u>

If given const. coef. eq'n w/ repeated root can use (*) w/o going through method of reduction

3.2 Higher order eq's, linear

$$y^{(n)} + p_1(x)\, y^{(n-1)} + \dots + p_n(x)\, y = f(x) \quad (*)$$

If $f \neq 0$ nonhomog., if $f \equiv 0$ homog.

nth order.

Ex: $y^{(4)} + y^{(3)} = y\sin(x) - \ln(x)$

linear 4th order.

Ex. & Uniqueness

$p_1, \dots, p_n, f$ cont. on interval I

If $a \in I$, $b_0, \dots, b_{n-1} \in \mathbb{R}$ then the IVP

$$\text{IVP}\begin{cases} y^{(n)} + p_1(x)\, y^{(n-1)} + \dots + p_n(x) y = f(x) \\ y(a) = b_0 \\ \vdots \\ y^{(n-1)}(a) = b_{n-1} \end{cases}$$

has a unique sol'n in all of I

So Ex:
$$\begin{cases} x\, y^{(3)} + \ln(x+1)\, y' = 2\sin(x) \\ y(1) = 1 \\ y'(1) = 2 \\ y''(1) = 3 \end{cases}$$

to find I where there is unique sol'n:

write: $y^{(3)} + \dfrac{\ln(x+1)}{x} y' = 2\,\dfrac{\sin(x)}{x}$

So $I = (0, \infty)$ largest int. where $\dfrac{\ln(x+1)}{x}$, $2\dfrac{\sin(x)}{x}$ cont. & $1 \in I$

Superposition: If $y_1, \dots, y_n$ sols to
$$y^{(n)} + p_1(x) y^{(n-1)} + \dots + p_n(x) y = 0 \quad (6)$$
homog!

then
$$y = c_1 y_1 + \dots + c_n y_n .$$
linear comb.

is also a sol'n.

Q: what are good building blocks that will produce all sol's of (6) as linear comb's?

Building blocks: lin. indep. sols.

Def'n: the fcts $f_1, \dots, f_n$ on interval I lin. independent if whenever we write
$$c_1 f_1(x) + \dots + c_n f_n(x) = 0 \text{ for all } x \in I$$
w/ c_j const. then necessarily $c_1 = c_2 = \dots = c_n = 0$

Otherwise: lin. dependent; at least one can be written as a linear comb. of the rest.

Ex: $4e^x$, $3e^{3x}$, $2e^x + 5e^{3x}$ lin. dependent:
$$(2e^x + 5e^{3x}) = \frac{1}{2}(4e^x) + \frac{5}{3}(3e^{3x})$$