Plan for today:
finish 3.2
start 3.3

Learning goals
1. Given n linearly independent solutions of an nth order linear ode, be able to find any other solution as a linear combination of them
2. Be able to check linear independence of n functions using the Wronskian
3. Be able to find n linearly independent solutions of a linear constant coefficient ODE whose characteristic equation has n distinct real roots.

Reminders/Announcements
1. Read the textbook!
2. Solutions to quiz 2 will be posted after lecture
3. Quiz 3 next Thursday (content will be announced later today)

Last time: Linear independence.

fcts $f_1, \dots, f_n$ on I lin. indep. if

$$c_1 f_1(x) + \dots + c_n f_n(x) = 0 \quad \forall \; x \in I$$

for all

$$\Rightarrow \quad c_1 = \dots = c_n = 0$$

Didn't see how to check lin. indep. $\rightarrow$ today.

Why care? Linearly indep. fcts $\rightarrow$ good building blocks for sols of linear ODE.

If $y_1, \dots, y_n$ are lin. indep. sols of

$$y^{(n)} + p_1(x) y^{(n-1)} + \dots + p_0(x) y = 0 \qquad (1)$$

then the general sol'n is

$$y = c_1 y_1 + \dots + c_n y_n$$

[sols of (1) form an n-dim'l vector space, $y_1, \dots, y_n$ are a basis]

Want to check lin. indep.

Wronskian: $y_1, \dots, y_n$ defined on I

$$W(y_1, \dots, y_n) = W(x) = \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & & \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & & y_n^{(n-1)}(x) \end{vmatrix}$$

n rows

n columns

determinant.

If $y_1, \dots, y_n$ are sols to

$$y^{(n)} + p_1(x) y^{(n-1)} + \dots + p_0(x) y = 0$$

on interval I then:

→ If $y_1, \dots, y_n$ lin. dep. then $W(x) \equiv 0$ on I

→ If $y_1, \dots, y_n$ lin. indep. then $W(x) \neq 0$ everywhere on I.

Ex: (HW in 3.2)

$$y^{(3)} - 6y'' + 11y' - 6y = 0 \quad (2)$$

3 sols given: $y_1 = e^x$, $y_2 = e^{2x}$, $y_3 = e^{3x}$

Want to solve IVP:

(2) w/ initial cond. $\begin{cases} y(0) = 1 \\ y'(0) = 2 \\ y''(0) = 3 \end{cases}$

Do we have 3 good building blocks?
Check linear indep.

$$W(x) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}$$

$$= e^x \cdot e^{2x} \cdot e^{3x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$$

$$= e^{6x}\left(1\cdot(18-12) - 1(9-4) + 1(3-2)\right)$$

$$= e^{6x}\,(6-5+1) = 2e^{6x} \neq 0$$

$\Rightarrow$ lin. indep.

So Any sol'n of 2

$$\boxed{y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}}$$

$$W(e^x, e^x, e^{2x}) = \begin{vmatrix} e^x & e^x & e^{2x} \\ e^x & e^x & 2e^{2x} \\ e^x & e^x & 4e^{2x} \end{vmatrix} = 0,$$

<u>Solve</u> IVP. $y(0)=1$, $y'(0)=2$, $y''(0)=3$

$$y(0) = c_1 + c_2 + c_3 = 1$$

$$y'(x) = c_1 e^x + c_2 \cdot 2e^{2x} + c_3 \cdot 3e^3$$

$$\Rightarrow y'(0) = c_1 + 2c_2 + 3c_3 = 2$$

$$y''(x) = c_1 e^x + c_2 \cdot 4 \cdot e^{2x} + c_3 \cdot 9 e^{3x}$$

$$\Rightarrow y''(0) = c_1 + 4c_2 + 9c_3 = 3$$

$$c_1 + c_2 + c_3 = 1$$

$$c_1 + 2c_2 + 3c_3 = 2$$

$$c_1 + 4c_2 + 9c_3 = 3$$

matrix form

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}}_{\text{det of this is } W(e^x, e^{2x}, e^{3x})(0)} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

initial cond.

So $W \neq 0$ means we can solve for (c_1, c_2, c_3). (Exercise) //
(sol'n below)

How did we find that e^x, e^{2x}, e^{3x} are sols to $y^{(3)} - 6y'' + 11y' - 6 = 0$?

const. coef. linear, homog.

Take char. eqn:

$$r^3 - 6r^2 + 11r - 6 = 0.$$

polynomial

If we can find 3 distinct real roots r_1, r_2, r_3 then $e^{r_1 x}, e^{r_2 x}, e^{r_3 x}$ will be lin. indep. sol's.

Tips for finding roots:

1. If $a_n r^n + \dots + a_1 r + a_0 = 0$, a_j all integers

and there is a rational root $\frac{p}{q}$ ← integers w/o common divisors > 1

then p divides a_0
q divides a_n.

Ex: $\frac{2}{3}$ ✓ $\frac{4}{6}$ ✗

$$r^3 - 6r^2 + 11r - 6 = 0$$

$$\begin{cases} a_3 = 1 \\ a_0 = -6 \end{cases}$$

If there is rational root $\frac{p}{q}$: p divides -6
q divides $1 \Rightarrow q = \pm 1$
$p = \underbrace{\pm 1, \pm 2, \pm 3, \pm 6}$

So if there is an integer root, one of ↑

Easiest possible root to check: 1 ; sum coef. & see if they add up to 0.

$$1^3 - 6 \cdot 1^2 + 11 \cdot 1 - 6 = 0$$

$$1 - 6 + 11 - 6 = 0$$

So 1 is a root!

Now $(r-1)$ divides $r^3 - 6r^2 + 11r - 6$, use long division to write

$$r^3 - 6r^2 + 11r - 6 = (r-1)\underbrace{Q(r)}_{\text{degree 2}}$$

degree 2, find roots. (sol'n below)

1. Want to solve for c_1, c_2, c_3 below.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Use row reduction (see p. 276-277)

Form augmented matrix:

$$\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \\ 1 & 4 & 9 & 3 \end{array}\right)$$

$$\begin{array}{c} \\ 2 - 1 \to 2 \\ 3 - 1 \to 3 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 8 & 2 \end{array}\right)$$

$$\begin{array}{c} \\ \\ 3 - 3 \cdot 2 \to 3 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & -1 \end{array}\right)$$

So
$$\begin{aligned} c_1 + c_2 + c_3 &= 1 \\ c_2 + 2c_3 &= 1 \\ 2c_3 &= -1 \end{aligned}$$

$\Rightarrow c_3 = -\frac{1}{2}, \quad c_2 = 2, \quad c_3 = -\frac{1}{2}$

So: sol'n $y = -\frac{1}{2} e^{x} + 2e^{2x} - \frac{1}{2} e^{3x}$ //

2. Long division:

$$\begin{array}{r} r^2 - 5r + 6 \\ r-1 \overline{) r^3 - 6r^2 + 11r - 6} \\ \oplus \; \underline{-r^3 + r^2 \qquad\qquad} \\ -5r^2 + 11r - 6 \\ \oplus \; \underline{5r^2 - 5r \qquad} \\ 6r - 6 \\ \oplus \; \underline{-6r + 6} \\ 0 \end{array}$$

So: $(r^3 - 6r^2 + 11r - 6) = (r-1)(r^2 - 5r + 6)$

By quadr. formula, roots are $r=2$, $r=3$.

So
$(r^3 - 6r^2 + 11r - 6) = (r-1)(r-2)(r-3)$
So $r=1$, $r=2$, $r=3$ are 3 distinct roots

and e^x, e^{2x}, e^{3x} are 3 lin. indep. sol's of

$$y^{(3)} - 6y'' + 11y' - 6 = 0$$