Plan for today

Finish 3.3

Learning goals:

1. Be able to solve n-th order constant coefficient linear equations whose characteristic equation has
   - A. n distinct real roots
   - B. A repeated real root
   - C. Complex roots
2. Know Euler ' s formula (very important!)

Reminders/announcements

1. Quiz this week, covers 2.5-3.3 the part discussed on Friday.
2. OH today 1-2, tomorrow 8.30-9.30 and 6-7 pm (Unusual time)
3. Read the textbook!

Last time: Const. coef. linear homog. eqs

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_0 y^{(0)} = 0 \quad (1)$$

Char. eqn:

$$a_n r^n + a_{n-1} r^{n-1} + \dots + a_0 \cdot 1 = 0 \quad (2)$$

r^0

If r root of (2) $\Rightarrow y = e^{rx}$ is sol'n.

Goal: find n lin. indep. building blocks, i.e. sols.

Cases: What do roots of (2) look like?

Case 1: n distinct real roots.

last time: found roots.

Ex: $y^{(3)} - 6y'' + 11y' - 6y = 0$

$r^3 - 6r^2 + 11r - 6 = 0$, roots $1, 2, 3$

gen. sol'n: $y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$

In general: $y = c_1 e^{r_1 x} + \dots + c_n e^{r_n x}$, $r_1, \dots, r_n$ are the roots of (2)

Case 2: Repeated real root.

$$y^{(4)} - 3y^{(3)} + 3y'' - y' = 0 \quad (3)$$

Char. eq:

$$r^4 - 3r^3 + 3r^2 - r = 0$$

$$r(r^3 - 3r^2 + 3r - 1) = 0$$

$\hookrightarrow$ $r = 0$ is a root!

$$r(r-1)^3 = 0$$

$\Rightarrow$ $r = 1$ repeated root, multiplicity 3

Can also check that 1 is a root (coef of $r^3 - 3r^2 + 3r - 1$ sum to 0)

Pascal's triangle:

1	$(a+b)^0$
1 1	$(a+b)^1$
1 2 1	$(a+b)^2$
1 3 3 1	$(a+b)^3$
1 4 6 4 1	
...	

$$(a+b)^2 = a^3 + 3a^2b + 3ab^2 + b^3$$

If r is repeated root, multiplicity k:
part of gen. sol'n corresponding to r is

$$(c_1 + c_2 x + c_3 x^2 + \dots + c_k x^{k-1}) e^{rx}$$

and $e^{rx}, xe^{rx}, \dots, x^{k-1}e^{rx}$ are lin. indep.

Gen. sol'n for (3):

$$y = c_1 \cdot \underset{\overset{\shortparallel}{1}}{e^0} + \underbrace{c_2 e^x + c_3 x e^x + c_4 x^2 e^x}_{\text{part cor. to } r=1.}$$

Ex: Char. eq'n: roots

3 $\rightarrow$ mult 2

5 $\rightarrow$ mult. 3

1 $\rightarrow$ mult. 1

$$y = c_1 e^{3x} + c_2 x e^{3x} + c_3 e^{5x} + c_4 x e^{5x} + c_5 x^2 e^{5x} + c_6 e^{x}$$

Case 3: Complex roots.

Ex: $\quad y'' + y = 0$

$r^2 + 1 = 0 \Rightarrow r^2 = -1$ no real roots, complex roots!

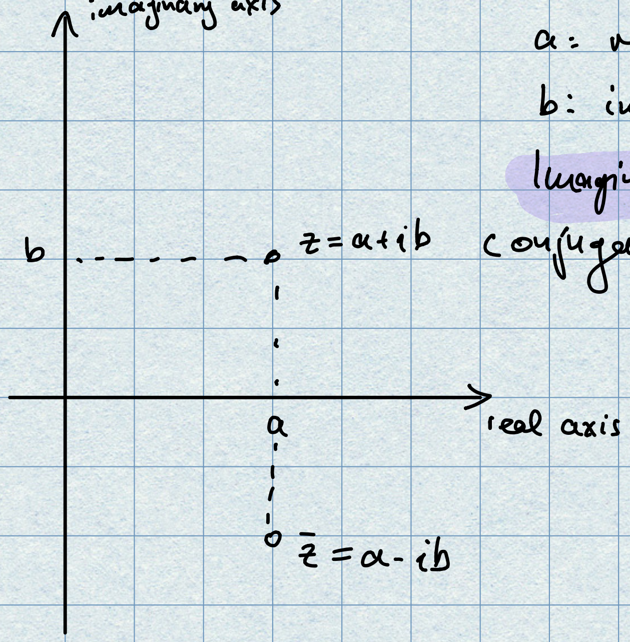
Complex number: $z = a + ib$, $a, b \in \mathbb{R}$, $i = \sqrt{-1}$

a: real pt of z, $a = Re(z)$

b: imaginary pt of z, $b = Im(z)$.

Imaginary pt of z is real!

conjugate of z: $\bar{z} = a - ib$



$r^2 = -1$: no real roots, $r = \pm i$ are complex roots.

Q: Can we make sense of e^{ix}, e^{-ix}, if yes, are they sol'ns of $y'' + y = 0$?

1: Make sense of $e^{(a+ib)x}$.

First look at $e^{i\theta}$, $\theta \in \mathbb{R}$

$$e^{i\theta} = \underbrace{\sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!}}_{\text{Taylor series}} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \dots$$

$i^2 = -1$
$i^3 = -i$
$i^4 = 1$

$$= 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \dots$$

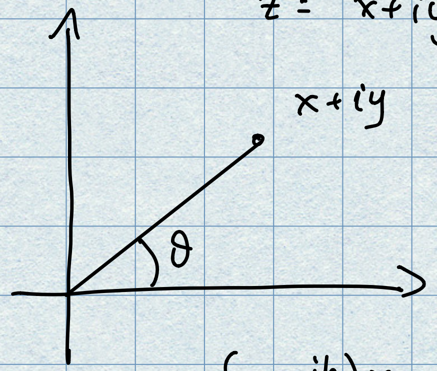
$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}\right)$$

$$\Rightarrow \boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta).}$$ Eulers formula

$$e^{a+ib} = e^a\left(\cos(b) + i\sin(b)\right)$$

Polar form of a cplx #:

$$z = x+iy = r\left(\frac{x}{r} + i\frac{y}{r}\right)$$
$$= r\left(\cos(\theta) + i\sin(\theta)\right)$$
$$= re^{i\theta}$$



Read ex: 5, § 3.3

$$y = e^{(a+ib)x} = e^{(ax)+i(bx)}$$
$$e^{a+ibx} = e^{ax}\left(\cos(bx) + i\sin(bx)\right)$$
$$= \underbrace{e^{ax}\cos(bx)}_{\text{Real pt}} + i\underbrace{e^{ax}\sin(bx)}_{\text{imaginary pt.}}$$

$y = e^{(a+ib)x}$ is a complex valued function.
a, b fixed. Input is the
real variable x.

$$x \in \mathbb{R} \longrightarrow \boxed{e^{(a+ib)x}} \xrightarrow{e^{(a+ib)x}} \in \mathbb{C}$$

$\mathbb{C} \to$ set of cplx #'s.

Next time: if $(a+ib)$ is a root of char. eq'n then $y = e^{(a+ib)x}$ is a sol'n.