Plan for today:
Finish 3.3
Start 3.4

Learning Goals

1. Be able to solve n-th order constant coefficient linear equations whose characteristic equation has
    A. n distinct real roots
    B. A repeated real root
    C. Complex roots
    D. Repeated complex roots
2. Be able to set up and solve a differential equation describing a spring-mass system in the presence or absence of damping. In this lesson we assume no external force.
3. In the case of no damping (free undated), be able to write the solution in the form $C\cos(\omega_0 t-\alpha)$
4. In the presence of damping, be able to determine whether the motion is underdamped, critically damped, or overdamped.

Announcements-Reminders

1. Read the textbook!
2. Quiz 3 tomorrow.

Last time: char. eq'n has complex roots

Ex: $y''+y=0 \Rightarrow r^2+1=0 \Rightarrow r=\pm i$

hoping: if $a+ib$ is a complex root, then $y=e^{(a+ib)x}$ is a solution.

made sense of this

$$y = e^{ax}\cos(bx) + i\, e^{ax}\sin(bx) \quad (1)$$

If $a+ib$ is a root then $y=e^{(a+ib)x}$ will satisfy original eq'n if $y'=(a+ib)e^{a+ib}$

Ex: want $(e^{ix})' = i e^{ix}$:

$$(e^{ix})'' = i^2 e^{ix} = -e^{ix} \Rightarrow (e^{ix})'' + e^{ix} = 0$$

satisfies $y''+y=0$

Dif'te real & imaginary pt of (1)

$$y' = \left(ae^{ax}\cos(bx) - e^{ax} b\sin(bx)\right) + i\left(ae^{ax}\sin(bx) + e^{ax}\cdot b\cos(bx)\right)$$

$$= (a+ib)e^{ax}\cos(bx) + i(a+ib)e^{ax}\sin(bx)$$

$i^2 = -1$

$$= (a+ib)\, e^{(a+ib)x} = (a+ib)y$$

→ If $y'' + p_1 y' + p_2 y = 0$, and $r = a \pm ib$ pair of cplx root of char. eq'n:

general sol'n

$$y = C_1 e^{ax}\cos(bx) + C_2 e^{ax}\sin(bx)$$

Ex; $y'' - 6y' + 13y = 0$

$$r^2 - 6r + 13 = 0$$

$$r = \frac{6 \pm \sqrt{36 - 4.13}}{2}$$

$$= \frac{6 \pm \sqrt{-16}}{2} = \frac{6 \pm 4i}{2}$$

$$= 3 \pm 2i$$

$e^{(3+2i)x}$, $e^{(3-2i)x}$ are sol's.

$$y = C_1 e^{(3+2i)x} + C_2 e^{(3-2i)x}$$ sol'n by superposition.

$$= C_1 e^{3x}\left(\cos(2x) + i\sin(2x)\right) + C_2 e^{3x}\left(\cos(2x) - i\sin(2x)\right)$$

$$= \underbrace{(c_1 + c_2)}\, e^{3x}\cos(2x) + \underbrace{i(c_1 - c_2)}\, e^{3x}\sin(2x)$$

$$\Rightarrow y = A e^{3x}\cos(2x) + B e^{3x}\sin(2x)$$

In general A, B cplx but if we have real initial conditions ($y(0), y'(0)$ real) then A, B real.

→ Pair of cplx roots w/ multiplicity k:

char. eq'n:

Ex: $(r^2 + 1)^k = 0 \rightarrow r = \pm i$, repeated k times

pair: $a \pm ib$ mult. k, part of general sol'n cor. to it is

$$\sum_{p=0}^{k-1} x^p e^{ax}(c_p \cos(bx) + d_p \sin(bx))$$

Sup.: $3 \pm 2i$ w/ mult. 3

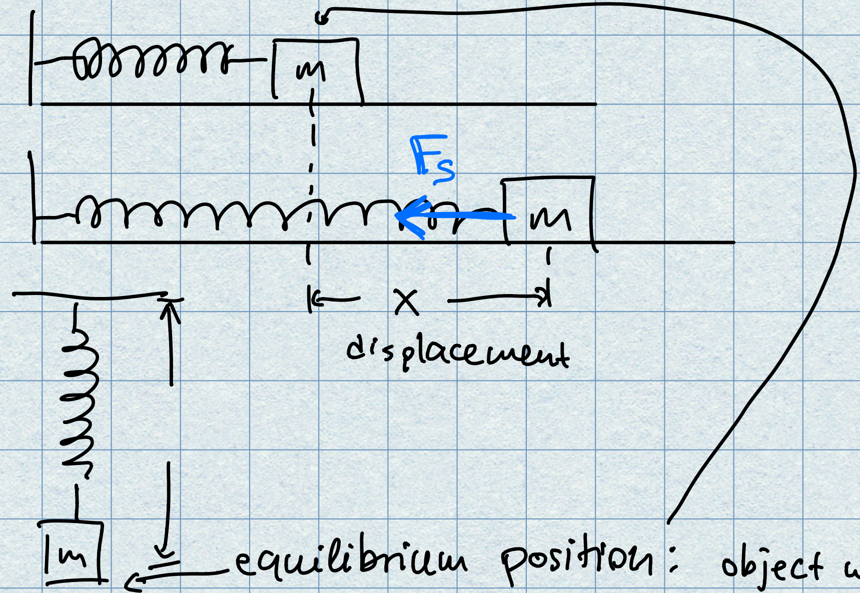
$$e^{3x}(c_1 \cos(2x) + c_2 \sin(2x)) + x e^{3x}(c_3 \cos(2x) + c_4 \sin(2x)) + x^2 e^{3x}(c_5 \cos(2x) + c_6 \sin(2x))$$

Exercise: Roots of char. eq'n: → 3 mult 1

$3 \pm i$ mult 1, $2 \pm 2i$ mult 3, 4 mult. 2.

Find general sol'n. [Sol'n at the end]

3.4. Mechanical Vibrations



equilibrium position: object will not move unless some initial velocity is given to it.

Spring: resists compression & stretching.

Hooke's Law: $$F_s = -k\,x$$

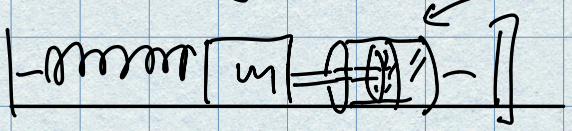
F_s: force of spring

k: spring const.

x: displacement from equil.

Additional forces:

→ Resisting motion (air resistance, friction, dashpot)



$$F_R = -cv = -c\frac{dx}{dt}$$

constant velocity

→ External force (your hand). $F_E = F(t)$

Goal: describe motion of object.

$$m\frac{d^2x}{dt^2} = F_{total} = F_S + F_R + F_E$$

$$\boxed{m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)}$$

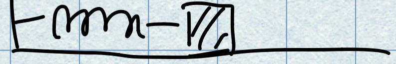
m, c, k const.

$F = 0$ → free motion

$F \neq 0$ → forced motion

$c = 0$ → undamped

$c \neq 0$ → damped.

Ex: Object of mass 2kg 

Spring stretched 2m by a force of 100 N.

Initial conditions $x_0 = 0$

$$v_0 = \left.\frac{dx}{dt}\right|_{t=0} = -8.$$

No damping $(c = 0)$

No external force.

$$m \frac{d^2x}{dt^2} + kx = 0$$

Find k: $\underbrace{-100\,N}_{\text{force, opposite to displ.}} = -\underbrace{k}\cdot\underbrace{2m}$

$k = 50\ N/m$

$$2\frac{d^2x}{dt^2} + 50x = 0 \Rightarrow \underline{\frac{d^2x}{dt^2} + 25x = 0}$$

<u>Char. eq'n</u>: $r^2 + 25 = 0 \Rightarrow r = \pm 5i$

According to today:

Gen. sol'n

$$x = A\cancel{e^{0t}}\cos(5t) + B\cancel{e^{0t}}\sin(5t)$$

$$= A\cos(5t) + B\sin(5t)$$

Find A, B using initial cond.

Exercise: Roots of char. eqn: → 3 mult 1
$3 \pm i$ mult 1, $2 \pm 2i$ mult 3, 4 mult. 2.

Find general sol'n.

Sol'n:

$$\begin{aligned} y = C_1 e^{3x} &+ e^{3x}\left(C_2 \cos(x) + C_3 \sin(x)\right) \\ &+ e^{2x}\left(C_4 \cos(2x) + C_5 \sin(2x)\right) \\ &+ x e^{2x}\left(C_6 \cos(2x) + C_7 \sin(2x)\right) \\ &+ x^2 e^{2x}\left(C_8 \cos(2x) + C_9 \sin(2x)\right) \\ &+ C_{10} e^{4x} + C_{11} x e^{4x} \end{aligned}$$