Plan for today:

Finish 3.4

Start 3.5

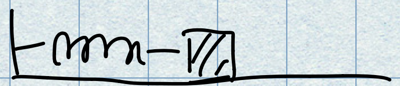
Learning Goals:

1. Be able to set up and solve a differential equation describing a spring-mass system in the presence or absence of damping. In this lesson we assume no external force.
2. In the case of no damping (free undated), be able to write the solution in the form $C\cos(\omega_0 t-\alpha)$
3. In the presence of damping, be able to determine whether the motion is underdamped, critically damped, or overdamped.
4. Be able to use the method of undetermined coefficients to solve non-homogeneous equations when the non-homogeneous term is a linear combination of products of polynomials, exponentials and trigonometric functions.

Reminders

1. Read the textbook!
2. Quiz should be graded by Monday.

Ex: Object of mass 2kg



find k ← { Spring stretched 2m by a force of 100 N.

Initial conditions $x_0 = 0$

$v_0 = \left.\frac{dx}{dt}\right|_{t=0} = -8.$

No damping $(c=0)$

No external force.

Gen. sol'n

$$x = A\cos(5t) + B\sin(5t)$$

Find A, B using initial cond.

$x(0)=0 \Rightarrow A = 0$

$x'(0) = -8 \Rightarrow \left. -5A\cancel{\sin(5t)}^{\,0} + 5B\cos(5t)\right|_{t=0} = -8$

$$\Rightarrow B = -\frac{8}{5}$$

Important: $A\cos(\omega_0 t) + B\sin(\omega_0 t)$

can be written as

$$C\cos(\omega_0 t - \alpha)$$

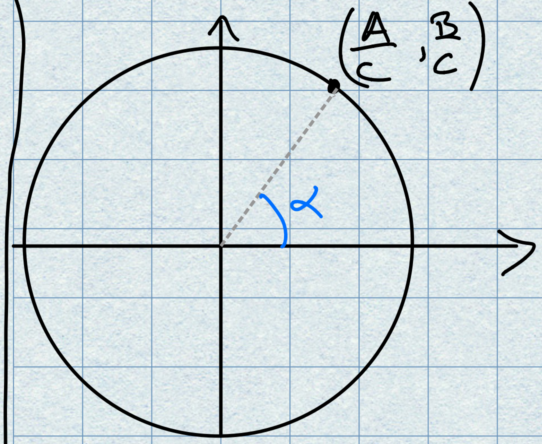
Why: $x = A\cos(\omega_0 t) + B\sin(\omega_0 t)$

$$= C\left(\frac{A}{C}\cos(\omega_0 t) + \frac{B}{C}\sin(\omega_0 t)\right)$$

where $\underbrace{C = \sqrt{A^2+B^2}}_{\downarrow \text{why?}}$ squares sum up to 1.

$$\left(\frac{A}{C}\right)^2 + \left(\frac{B}{C}\right)^2 = \frac{A^2}{A^2+B^2} + \frac{B^2}{A^2+B^2} = 1$$

$\left(\frac{A}{C}, \frac{B}{C}\right) \rightarrow$ pair on unit circle.



There is angle α:

$$\cos(\alpha) = \frac{A}{C}, \quad \sin(\alpha) = \frac{B}{C}$$

$$\alpha = \begin{cases} \tan^{-1}\left(\frac{B}{A}\right) & A > 0, B > 0 \\ \pi + \tan^{-1}\left(\frac{B}{A}\right) & A < 0 \\ 2\pi + \tan^{-1}\left(\frac{B}{A}\right) & A > 0 \\ & B < 0 \end{cases}$$

$$x = C\left(\cos(\alpha)\cos(\omega_0 t) + \sin(\alpha)\sin(\omega_0 t)\right)$$

$$\overset{\text{trig. id.}}{=} C\cos(\omega_0 t - \alpha). \qquad //$$

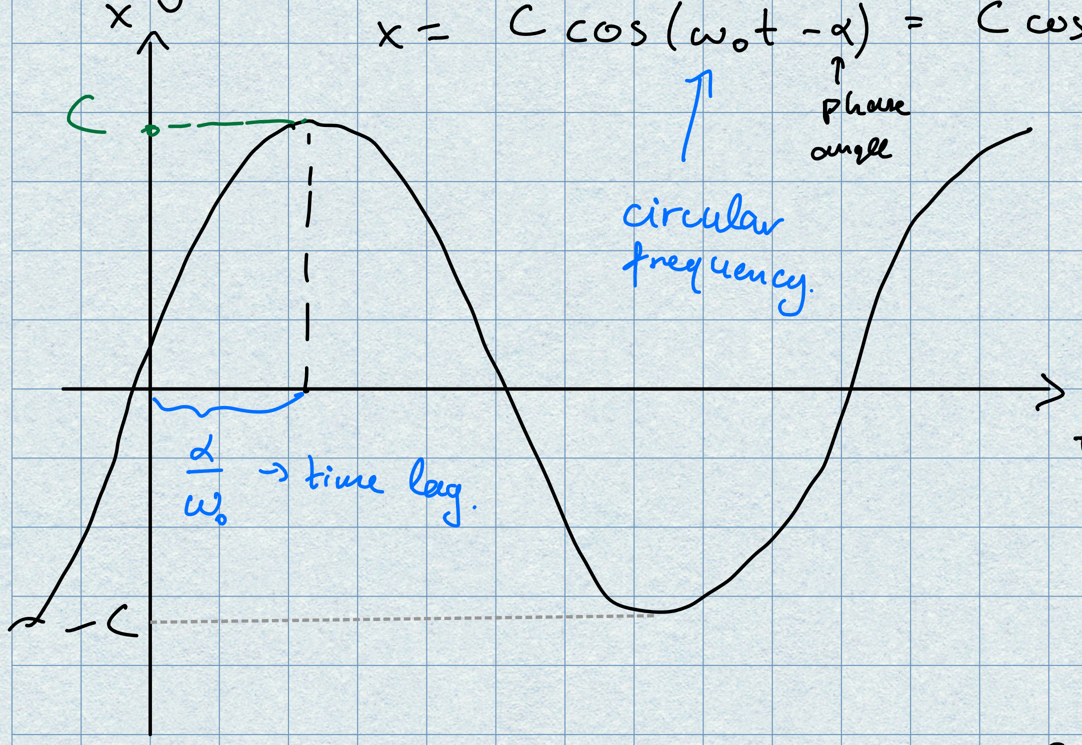
Why bother:

amplitude

$$x = C\cos(\omega_0 t - \alpha) = C\cos\left(\omega_0\left(t - \frac{\alpha}{\omega_0}\right)\right)$$

phase angle

circular frequency



In our example: $x = \underbrace{0}_{A} \cdot \cos(5t) \underbrace{- \frac{8}{5}}_{B} \sin(5t)$

$$C = \sqrt{A^2 + B^2} = \frac{8}{5} > 0$$

$$\left.\begin{aligned} \cos(\alpha) &= 0 \\ \sin(\alpha) &= \frac{B}{C} = \frac{-\frac{8}{5}}{\frac{8}{5}} = -1. \end{aligned}\right\} \Rightarrow \alpha = \frac{3\pi}{2}$$

So: $$x = \frac{8}{5}\cos\left(5t - \frac{3\pi}{2}\right)$$

<u>Damping:</u> $c > 0$

$$m x'' + c x' + k x = 0$$

no external force.

Char. eq: $m r^2 + c r + k = 0$ (I)

$$\Rightarrow \quad r = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

Behavior depends on discr. $c^2 - 4km$

Critical damping: $c_{cr}^2 - 4km = 0 \Leftrightarrow c_{cr} = \sqrt{4km}$

→ If $c > c_{cr}$, $c^2 - 4km > 0$ (check!)

(I) has 2 real roots, < 0. gen. sol

$$x = A e^{(-c + \sqrt{c^2 - 4km})/2m \cdot t} + B e^{(-c - \sqrt{c^2 - 4km})/2m \cdot t}$$

negative (check!)

overdamped motion.

→ $c = c_{cr}$ repeated root $r = -\frac{c}{2m}$

$$x = A e^{-\frac{c}{2m} t} + B t e^{-\frac{c}{2m} t}$$

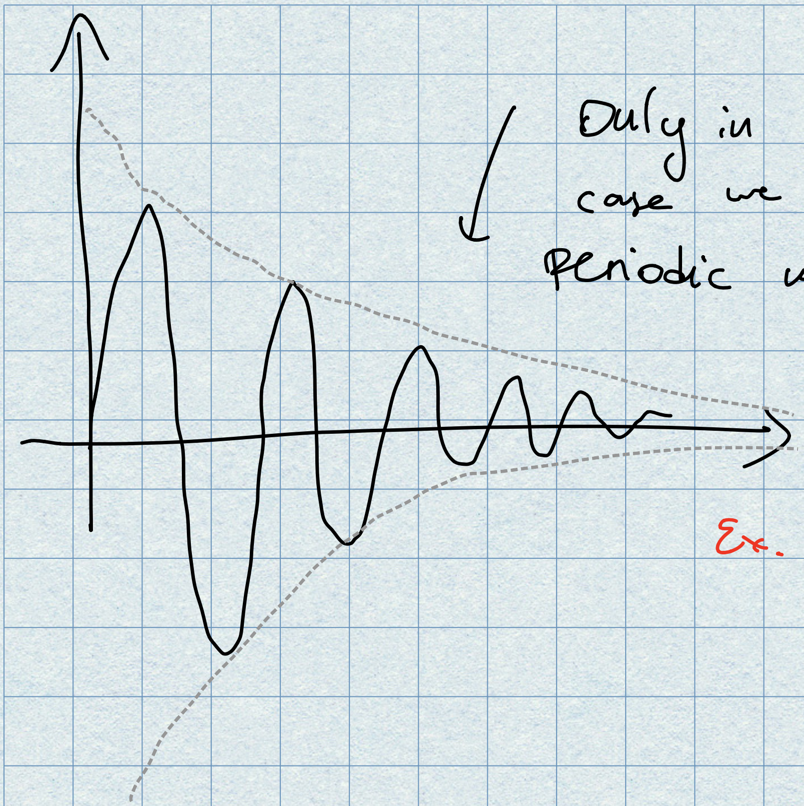
critially damped

→ $c < c_{cr}$ underdamped motion.

$$x(t) = \underbrace{A e^{-\frac{c}{2m} t} \cos(\omega_1 t) + B e^{-\frac{c}{2m} t} \sin(\omega_1 t)}$$

$$\left(\omega_1 = \frac{\sqrt{4km - c^2}}{2m} \right)$$

$$\underbrace{e^{-\frac{c}{2m} t} C}_{\text{time-changing amplitude.}} \cos(\omega_1 t - \alpha)$$



Only in underdamped case we have pseudo-periodic motion

Ex. at the end.

3.5 Intro to method of undetermined coef.

What does this do?
When does it apply?
How do we use it? → Monday.

So far: seen homog. eq's, const. coef.

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_0 y = 0 \quad (2)$$

Ex.: $y^{(3)} + 3y'' + 5y' - 2y = 0$

What if we have non-homog. eq'n?

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_0 y = f(x) \quad (3)$$

Key idea: Any sol'n of 3 can be written as $y = y_c + y_p$

"complementar sol'n" i.e. gen. sol'n of 2

particular sol'n of 3

Given 3, goal is to find → general sol' of 2
→ particular sol' of 3

Ex: $y'' - 4y' = \sin(x)$ 4

gen sol'n of $y'' - 4y' = 0$: $r^2 - 4r = 0$

So: $y = A + Be^{4x}$

↑ gen. sol'n of associated homog. eq'n.

Want one particular sol'n of 4

Turns out (Monday)

$$y_p = \frac{4}{17}\cos(x) - \frac{1}{17}\sin(x)$$

So: general sol'n to 4 is

$$y = A + Be^{4x} + \frac{4}{17}\cos(x) - \frac{1}{17}\sin(x)$$

Ex. of underdamped motion

Same ex. as before: $k = 50\,N/m$

$m = 2\,kg$

$x_0 = 0$

$v_0 = -8$

now w/ damping $c = 12$

Eqn:

$$2x'' + 12x' + 50x = 0$$

$$r^2 + 6r + 25 = 0 \Rightarrow r = -3 \pm 4i$$

Gen. sol'n:

$$x(t) = Ae^{-3t}\cos(4t) + Be^{-3t}\sin(4t)$$

Check:

$$x(0) = 0 \Rightarrow A = 0$$

$$x'(0) = -8 \Rightarrow B = -2$$

$$x(t) = -2e^{-3t}\sin(4t)$$

$$= 2e^{-3t}\cos\left(4t - \frac{3\pi}{2}\right)$$

Note that the pseudo-frequency is smaller than in the undamped case.