Plan for today:
3.5

Learning goals
1. Know when the method of undetermined coefficients applies and be able to use it
2. Be able to find building blocks for the non homogeneous term and its derivatives.
3. Know how to handle the case where there is part of the complementary solution appearing as a building block for the non homogeneous term or its derivatives.

Announcements/Reminders
1. Quiz grades will be posted later today
2. Read the textbook!
3. Computer Project 1 due Friday.

<u>Friday:</u> Non-homog. eqs.

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_0 y = f(x) \quad (1)$$

gen. sol'n: $y = y_c + y_p$

y_c: sol'n to $y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_0 y = 0 \quad (2)$

y_p: particular sol'n of (1)

<u>Undetermined coefficients</u>: method for finding y_p.

<u>When it works:</u> 1. Const. coef.

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_0 y = f(x)$$

const.

2. f "nice": finite sum of products of
1. exponentials
2. polynomials $\quad (x^k + p_1 x^{k-1} + \dots + p_0)$ $\quad k \geq 0$
3. $\cos(kx)$, $\sin(kx)$

Ex:

$$y'' + 4y = \cos(x)e^{x} + e^{3x}(x^2+x) + \sin(x)\cos(5x)\,e^{5x}(x^5+4)$$

$$y^{(3)} + 2y' + y = 3 + \sin(x) + e^{-x}$$

polynomial

Non-ex.

$$y'' + \underbrace{x}_{\text{not const. coef.}}y' + y = \cos(x)$$

$$y'' + y' = \frac{1}{x}$$

$$y''' + 2y' = \tan(x) = \frac{\sin(x)}{\cos(x)}$$

ratios not allowed

Idea: come up w/ good guess for y_p.
Form a linear comb. of function on RHS and all of its derivatives.

Ex: $y'' - 4y' = \sin(x)$ (3)

Last time: saw that $y_c = C_1 + C_2 e^{4x}$

$(\sin(x))' = \cos(x)$

$(\sin(x))'' = -\sin(x)$

$(\sin(x))^{(3)} = -\cos(x)$

$(\sin(x))^{(4)} = \sin(x)$

all derivatives of any order of $\sin(x)$ are built from two building blocks: $\sin(x)$, $\cos(x)$.

Take: $y_p = A\cos(x) + B\sin(x)$, try to find A, B.

Plug in to 3:

$$(-A\cos(x) - B\sin(x)) - 4(-A\sin(x) + B\cos(x)) = \sin(x)$$

collect terms =

$$\underbrace{(-A - 4B)\cos(x) + (-B + 4A - 1)\sin(x)}_{\text{linear comb. of } \cos(x),\ \sin(x)} = 0$$

lin. indep: $c_1 f_1(x) + c_2 f_2(x) = 0$ for all x then $c_1 = c_2 = 0$

$$\begin{cases} -A - 4B = 0 \\ -B + 4A - 1 = 0 \end{cases} \Rightarrow \begin{cases} B = -\frac{1}{17} \\ A = \frac{4}{17} \end{cases}$$

$$y_p = \frac{4}{17}\cos(x) - \frac{1}{17}\sin(x).$$

gen. sol:

$$y = c_1 1 + c_2 e^{4x} + \underbrace{\frac{4}{17}\cos(x) - \frac{1}{17}\sin(x)}_{\text{doesn't contribute free parameters.}}$$

2 free parameters

Check that satisfies $y'' - 4y' = \sin(x)$

! Note: none of building blocks $\cos(x)$, $\sin(x)$ appeared in the complementary sol'n.
("no duplication")

What if that isn't the case:

Ex: $y'' - 2y' + y = e^x$

Step 1: find complementary sol'n.

Char: $r^2 - 2r + 1 = 0 \Rightarrow r = 1$ repeated.

$$y_c = c_1 e^x + c_2 x e^x$$

Step 2: Find building blocks & make good guess.

building block: e^x

$$\hat{y} = Ae^x$$

what happens: $\hat{y}'' - 2\hat{y}' + \hat{y} = Ae^x - 2Ae^x + Ae^x = 0 \neq e^x$

our guess is not good!

If any term in linear comb. we formed is part of y_c then multiply the whole linear comb. by smallest integer power of x so that there is no duplication,

$$y = x^s A e^x \qquad s = ?$$

$s=0$ ×

$s=1$ ×

$s=2$ ✓

<u>Guess:</u> $y = Ax^2e^x$

<u>plug in:</u> $y'' - 2y' + y = e^x$

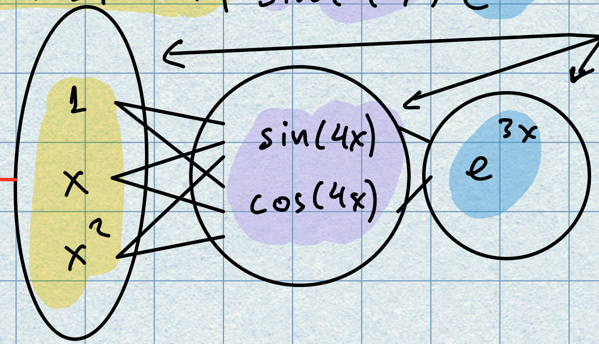
...

find A.

$\boxed{\text{sol'n at the end!}}$

<u>Ex:</u> How to find building blocks of fct & derivatives.

$$f(x) = (x^2+3x+1)\sin(4x)e^{3x}$$



building blocks for each factor in simplest form

$$\left.\begin{aligned}(x^2+3x+1)' &= 2x+3\\ (x^2+3x+1)'' &= 2\\ (x^2+3x+1)''' &= 0\end{aligned}\right\}$$ Building blocks for x^2+3x+1: $1, x, x^2$

all combinations

$$A\cdot 1\cdot\sin(4x)e^{3x} + B\cdot 1\cdot\cos(4x)e^{3x}$$
$$+ C\cdot x\sin(4x)e^{3x} + D\cdot x\cos(4x)e^{3x}$$
$$+ E\,x^2\sin(4x)e^{3x} + F\,x^2\cos(4x)e^{3x}. \quad //$$

Sol'n:

$y_p = Ax^2e^x$ <

$y_p'' = 2Ae^x + 4xe^x + Ax^2e^{2x}$

$y_p' = 2Axe^x + Ax^2e^x$

$y_p = Ax^2e^x$

$$y_p'' - 2y_p' + y_p = \cancel{Ax^2e^{2x}} + \cancel{4xAe^x} + 2Ae^x - \cancel{4Axe^x} - \cancel{2Ax^2e^x} + \cancel{Ax^2e^x} = e^x$$

$\Rightarrow 2Ae^x = e^x \Rightarrow A = \frac{1}{2}$

$\Rightarrow y_p = \frac{1}{2}x^2e^x$

General sol'n:

$$y = c_1e^x + c_2xe^x + \frac{1}{2}x^2e^x$$