Plan for today:
Finish 3.5
Start 3.6

Learning Goals:
1. Be able to use undetermined coefficients, with or without duplication
2. Be able to use variation of parameters: make sure to bring your equation in standard form first
3. Know when each method applies
4. Be able to use undetermined coefficients to solve problems in the context of mechanical vibrations
5. Be familiar with the notation of differential operators (originally appeared in § 3.3)

Reminders:
1. Read the textbook

Undetermined Coefficients. (for non-homog. eq's w/ specific structure)

One more example.

$$\underbrace{y'' - 6y' + 13y}_{\text{const. coef.}} = \underbrace{\overbrace{xe^{3x}\sin(2x)}^{\text{sum}} + \overbrace{\cos(x)}}_{\text{sum of products of polyn., exp, trig.}} \quad (*)$$

We can use undet. coef.

Note: Solve separately
$$\begin{cases} y'' - 6y' + 13y = xe^{3x}\sin(2x) & (1) \\ y'' - 6y' + 13y = \cos(x) & (2) \end{cases}$$

Spoxe $y_1 \to$ sol'n to (1)
$y_2 \to$ sol'n to (2)

Then: $y = y_1 + y_2$ solves $(*)$ [check!]

Look at (2):

Step 1: compl. sol'n.

$$r^2 - 6r + 13 \Rightarrow r = 3 \pm 2i$$

$$y_c = C_1 e^{3x}\cos(2x) + C_2 e^{3x}\sin(2x)$$

Step 2: Building blocks for $\cos(x)$ & its derivatives.

$$y = A\cos(x) + B\sin(x)$$ (looking for A, B)

Step 3: Do we have duplication? No! none of $\cos(x)$, $\sin(x)$ appear in y_c.

So: $y = A\cos(x) + B\sin(x)$ is a good guess.

Step 4: Plug in $y = A\cos(x) + B\sin(x)$ into

$$y'' - 6y' + 13y = \cos(x)$$

& find A, B. [Sol'n at the end]

Look at (1)

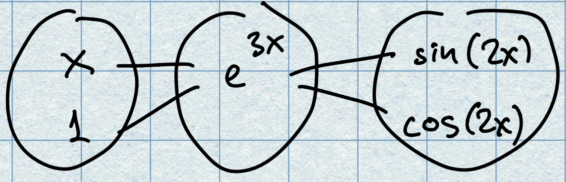
$$y'' - 6y' + 13y = x e^{3x}\sin(2x)$$

Step 1: Compl. sol'n:

$$y_c = C_1 e^{3x}\cos(2x) + C_2 e^{3x}\sin(2x)$$

Step 2: Building blocks for $x e^{3x}\sin(2x)$ & derivatives

$x \qquad e^{3x} \qquad \sin(2x)$



$$\tilde{y} = A\, x e^{3x} \sin(2x) + B\, x e^{3x} \cos(2x)$$
$$+ C \boxed{e^{3x} \sin(2x)} + D \boxed{e^{3x} \cos(2x)}$$

Step 3: Duplication? Yes! So not a good guess.

$$y = x\tilde{y} = A\, x^2 e^{3x} \sin(2x) + B\, x^2 e^{3x} \cos(2x) + C\, e^{3x} x \sin(2x) + D\, e^{3x} x \cdot \cos(2x)$$

Step 4: plug in, find $A, B, C, D \rightarrow$ soln y_1.

Why we split: 1 has duplication, 2 doesn't.

If we tried to treat as one instead of splitting:

Guess:

coming from 2.

$$y = \tilde{A} \cos(x) + \tilde{B} \sin(x)$$
$$+ A\, x e^{3x} \sin(2x) + B\, x e^{3x} \cos(2x)$$
$$+ C \boxed{e^{3x} \sin(2x)} + D \boxed{e^{3x} \cos(2x)}$$

coming from 1

Work w/ xy bec. of duplication.

$\hookrightarrow$ $\underline{x \cos(x)}$, $\underline{x \sin(x)}$

use product rule in computation $\rightarrow$ harder.

Undet. coef. $\to$ restrictive method

$$a_n y^{(n)} + \dots + a_0 y = f(x)$$

1. const. coef.
2. special form of f

A more general method: Variation of Parameters.

$$y'' + P(x)y' + Q(x)y = f(x) \quad (**)$$

can depend on x. any reasonable function.

A part. sol'n for is

$$y_p = -y_1(x)\int \frac{y_2(x)f(x)}{W(y_1,y_2)}\,dx + y_2(x)\int \frac{y_1(x)f(x)}{W(y_1,y_2)}\,dx$$

y_1, y_2 2 lin. indep. sols of $(**)$

$$W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$ (bec. y_1, y_2 lin. indep.)

Ex: Popular final q'n.

$$\underbrace{(x^2-1)\,y'' - 2x\,y' + 2y}_{\text{on } I=(-1,1)} = x^2-1$$

linear non-homog

$y_1 = x$

$y_2 = x^2+1$

Can we use undet. coef? No! (not const. coef.)

! Divide by (x^2-1) to get to standard form.

$$y'' - \frac{2x}{x^2-1} y' + \frac{2}{x^2-1} y = 1.$$

$$W(y_1, y_2) = \begin{vmatrix} x & x^2+1 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 - 1$$
$$= x^2 - 1$$

$$y_p = - x \int \frac{(x^2+1) \cdot 1}{x^2-1} dx + (x^2+1) \int \frac{x \cdot 1}{x^2-1} dx$$

$$= -x \int 1 + \frac{2}{x^2-1} dx + (x^2+1) \int \frac{x}{x^2-1} dx$$

$$= -x \left(x + \ln \left(\frac{1-x}{1+x} \right) \right) + (x^2+1) \frac{1}{2} \ln (x^2-1)$$

Limitation: integrals might be harder to compute.

For ② undet. coef is easier, though var. of parameters works.

but $y'' + 2y + 3y = \tan(x)$ → var. of param. only bec. $\tan(x)$ not good for undet coef.

Sol's :

2 : $y_2 = A\cos(x) + B\sin(x)$

building blocks for all derivatives of $\cos(x)$

$$y_2'' - 6y_2' + 13y_2 = \cos(x)$$

$$-A\cos(x) - B\sin(x)$$
$$- 6(-A\sin(x) + B\cos(x))$$
$$+ 13(A\cos(x) + B\sin(x)) = \cos(x)$$

$$(-A - 6B + 13A - 1)\cos(x)$$
$$+ (-B + 6A + 13B)\sin(x) = 0$$

$$\Rightarrow \begin{cases} 6B - 12A = 1 \\ 6A + 12B = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 6B + 24B = 1 \\ A = -2B \end{cases} \Rightarrow \begin{cases} B = \frac{1}{30} \\ A = -\frac{1}{15} \end{cases}$$

$$y_1 = -\frac{1}{15}\cos(x) + \frac{1}{30}\sin(x)$$

1 :

Plug in

$$y = x\tilde{y} = A x^2 e^{3x}\sin(2x) + B x^2 e^{3x}\cos(2x)$$
$$+ C e^{3x} x\sin(2x) + D e^{3x} x\cdot\cos(2x)$$

into $y'' - 6y' + 13y = xe^{3x}\sin(2x)$

Using Mathematica:

$$(2B + 4C)e^{3x}\cos(2x) + 8A\,xe^{3x}\cos(2x)$$
$$+ (2A - 4D)e^{3x}\sin(2x) - 8B\,xe^{3x}\sin(2x)$$
$$= xe^{3x}\sin(2x)$$

$$\Rightarrow \begin{cases} -8B = 1 \\ A - 2D = 0 \\ 8A = 0 \\ 2B + 4C = 0 \end{cases} = \begin{cases} B = -\frac{1}{8} \\ D = 0 \\ A = 0 \\ C = \frac{1}{16} \end{cases}$$

So

$$y_p = \frac{1}{16} e^{3x} x \sin(2x) - \frac{1}{8} x^2 e^{3x} \cos(2x)$$

General sol'n to (*)

$$y = C_1 e^{3x} \cos(2x) + C_2 e^{3x} \sin(2x)$$

$$-\frac{1}{15} \cos(x) + \frac{1}{30} \sin(x)$$

$$+ \frac{1}{16} e^{3x} x \sin(2x) - \frac{1}{8} x^2 e^{3x} \cos(2x)$$