Plan for today:
§3.6

Learning goals
1. Be able to use undetermined coefficients to solve problems in the context of mechanical vibrations
2. Be able to identify whether there is resonance (undamped forced motion) or practical resonance (damped forced motion)
3. Be familiar with the notation of differential operators (originally appeared in §3.3)

Reminders:
1. Fill out survey
2. Read the textbook

HW 18 #29

Notation of dif'l op'tors (originally in § 3.3)

y' can think of as Dy

operator: function which eats functions, outputs functions.

$$y \to \boxed{D} \to y' = Dy$$

$$x^2 \to \boxed{D} \to 2x \to \boxed{D} \to 2$$

Dif. eqn with const. coef.

$$3y'' + 4y' + y = 0$$

$$3D^2 y + 4Dy + y = 0$$

operator

$$Ly = 0, \quad L = 3D^2 + 4D + 1$$

multiply by 1.

dif. eqn in operator notation

Advantage: (const. coef.) can read off characteristic eq'n easily : replace D w/ r.

$$(3D^2 + 4D + 1)y = 0$$

Char. eq'n: $3r^2 + 4r + 1 = 0$

Ex: $(D-1)(D^2+1)y = 0$

char. eq'n: $(r-1)(r^2+1) = 0 \Rightarrow r = 1,\ r = \pm i$

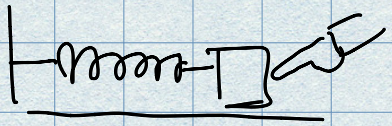
algebraic eq'n.

$$(D^3 + D - D^2 - 1)y = 0$$ ← dif eq'n in operator form

$$y''' - y'' + y' - y = 0$$

Forced oscillations

mass attatched to spring w/ external force, possible damping



assume periodic external force

$$\underset{\text{mass}}{m}x'' + \underset{\text{damping const.}}{c}x' + \underset{\text{spring const.}}{k}x = \overbrace{F_0\cos(\omega t)}$$

(undamped forced motion).

$$mx'' + kx = F_0\cos(\omega t)$$

$$x'' + \frac{k}{m}x = \frac{F_0}{m}\cos(\omega t)$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

Ex: $x'' + 4x = 5\sin(3t)$

$x(0) = x'(0) = 0$

1. compl. sol'n: char. eq'n $r^2+4=0$

$r = \pm 2i$

$$x_c = c_1\cos(2t) + c_2\sin(2t)$$

2. guess for x_p:

$$x_p = A\cos(3t) + B\sin(3t)$$

3. Duplication? No! good guess!

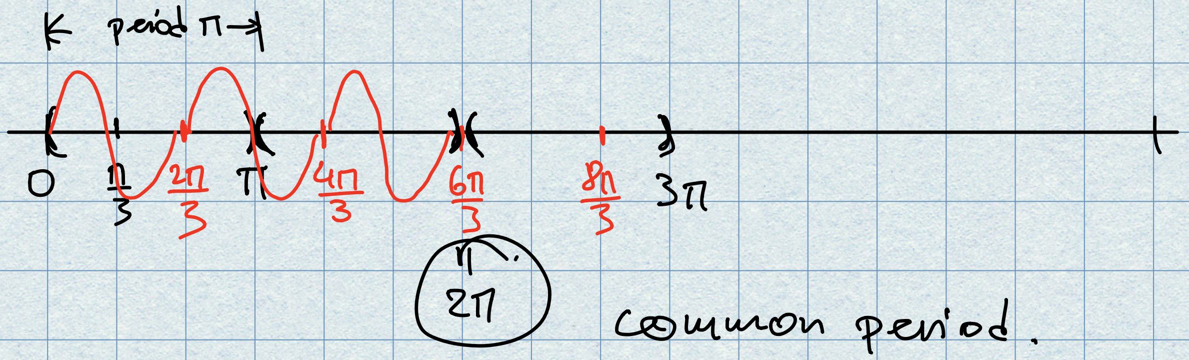
4. Find A, B by plugging in. [sol'n at the end]

Sol'n: $$x = c_1\overbrace{\cos(2t) + c_2\sin(2t)}^{\text{period } \pi}$$
$$+ \underbrace{A\cos(3t) + B\sin(3t)}_{\text{known} \qquad \text{period } \frac{2\pi}{3}}$$

5. Find c_1, c_2 to match initial conditions.

$$\cos(\omega_0 t) \xrightarrow{\text{period}} T = \frac{2\pi}{\omega_0}$$

The whole sol'n x is periodic!



common period.

If $\frac{\omega_0}{\omega} = \frac{p}{q}$, p, q integers then

period of $A\sin(\omega t) + B\cos(\omega t) + C\cos(\omega_0 t) + D\sin(\omega_0 t)$

is $\frac{2\pi q}{\omega}$

$\omega_0 = 2$

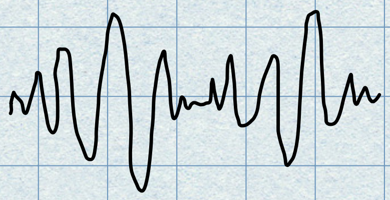
$\omega = 3$

$$\frac{\omega_0}{\omega} = \frac{\overset{=p}{2}}{\underset{=q}{3}} = \frac{2\pi\cdot 3}{3} = 2\pi$$

period $\frac{2\pi\cdot 2}{2} = 2\pi$

Had: $\omega_0 = 2 = \sqrt{\frac{k}{m}}$

took $\omega = 2.1 \to$ amplitude periodic in time



When $\omega \sim \omega_0$ but not equal: beats.
(read in book).

If $\omega = \omega_0$: amplitude increased unboundedly: resonance.

Ex: $x'' + 4x = 5\cos(2t)$

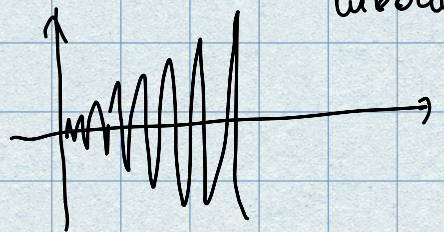
1. $x_c = c_1\cos(2t) + c_2\sin(2t)$

2. $\tilde{x}_p = A\cos(2t) + B\sin(2t)$

3. Duplication? Yes!

$$x_p = A\,t\cos(2t) + B\,t\sin(2t)$$

unbounded increase in amplitude



Forced Damped Motion

Saw: free damped motion: $mx'' + cx' + kx = 0$

(1) Overdamped: $x = c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$, $r_1, r_2 > 0$

(2) Critically damped: $x = (c_1 + c_2 t)e^{-rt}$, $r > 0$

(3) Underdamped: $x = Ce^{-rt}\cos(\omega_1 t - \alpha)$ $r > 0$

Exponentially fast decay as $t \to \infty$.

$\to$ $mx'' + cx' + kx = F_0\cos(\omega t)$. (forced)

Undetermined coef.:

1. Compl soln. looks like (1), (2) or (3)
2. guess part. soln

$$x_p = A\cos(\omega t) + B\sin(\omega t)$$

3. No duplication!

General soln: $x = \underbrace{x_c}_{\substack{\text{transient sol'n,}\\ \text{decays exp. fast.}\\ \text{one of (1), (2), (3)}}} + \underbrace{A\cos(\omega t) + B\sin(\omega t)}_{\text{steady periodic sol'n}}$

Sol'n of Ex:

$$x'' + 4x = 5\sin(3t)$$

$$x_c = c_1 \cos(2x) + c_2 \sin(2x)$$

Guess for p. sol'n: $x_p = A\cos(3t) + B\sin(3t)$

Plug in:

$$x_p'' + 4x_p = 5\sin(3t)$$

$$\Rightarrow \quad -9A\cos(3t) - 9B\sin(3t) + 4A\cos(3t) + B\sin(3t) = 5\sin(3t)$$

$$\Rightarrow \begin{cases} -9B + B = 5 \Rightarrow B = -\frac{5}{8} \\ -9A + 4A = 0 \Rightarrow A = 0 \end{cases}$$

So $x_p = -\frac{5}{8}\sin(3t)$

So: $x = c_1 \cos(2x) + c_2 \sin(2x) - \frac{5}{8}\sin(3x)$.

Want $x(0) = 0$, $x'(0) = 0$

So $c_1 = 0$

$$x' = -2c_1 \sin(2x) + 2c_2 \cos(2x) - \frac{15}{8}\cos(3x)$$

$$\Rightarrow \quad 2c_2 - \frac{15}{8} = 0 \Rightarrow c_2 = \frac{15}{16}$$

So $x = \frac{15}{16}\sin(2x) - \frac{5}{8}\sin(3x)$ //