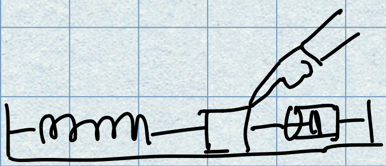
Plan for today:
Finish 3.6
Start 4.1

Learning goals
1. Be able to find if there is practical resonance.
2. Be able to set up equations for mass-spring systems
3. Be able to turn a high order equation into a system and vice versa
4. Turn a high order system into a first order system with more equations

Announcements/Reminders:
1. Quiz 4 this week on Sections 3.3-3.6
2. Read the textbook!
3. OH today and tomorrow
4. HW due tomorrow and this Friday

From Friday:



Forced Damped Motion.

Saw: free damped motion: $mx'' + cx' + kx = 0$

1 Overdamped: $x = c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$, $r_1, r_2 > 0$

2 Critically damped: $x = (c_1 + c_2 t) e^{-rt}$, $r > 0$

3 Underdamped: $x = C e^{-rt} \cos(\omega_1 t - \alpha)$ $r > 0$

Exponentially fast decay as $t \to \infty$.

$\to$ $mx'' + cx' + kx = \underbrace{F_0 \cos(\omega t)}_{\text{periodic force}}$ (forced)

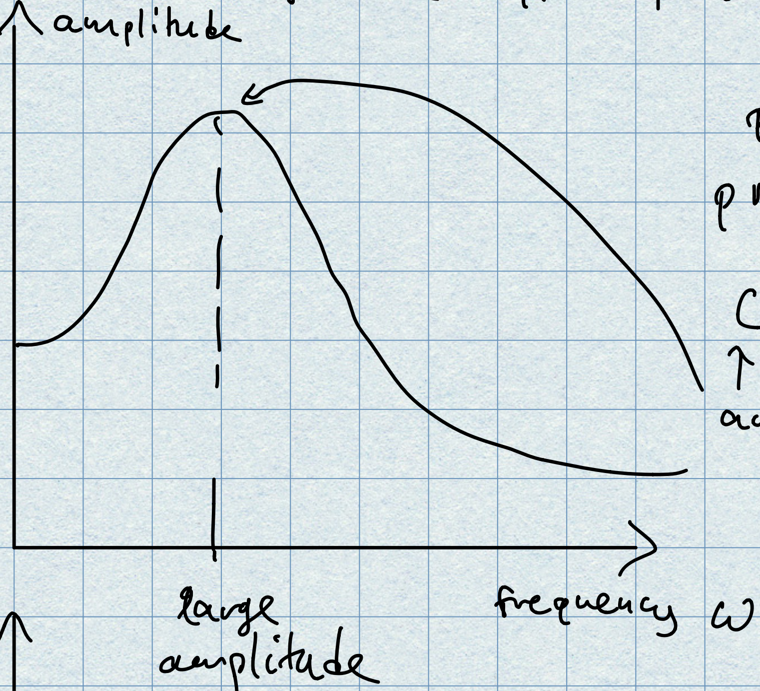
Undetermined coef:

1. Comp! sol'n. looks like 1, 2 or 3
2. guess part. soln
$$x_p = \underbrace{A\cos(\omega t) + B\sin(\omega t)}$$
3. No duplication! ↳ doesn't overlap w/ 1, 2, 3

General sol'n: $x = \underbrace{x_c}_{\substack{\text{transient sol'n,} \\ \text{decays exp. fast.} \\ \text{one of (1), (2), (3)}}} + \underbrace{A\cos(\omega t) + B\sin(\omega t)}_{\text{steady periodic sol'n}}$

<u>On Friday</u>: As $\omega \to \omega_0$ in undamped case amplitude grows. When $\omega = \omega_0$ then amplitude unbounded in time (resonance)

Here: amplitude will depend on frequency:



possibility 1: practical resonance

$C(\omega)$ has critical pt.
amplitude

no practical resonance.

Ex: $x'' + x' + 5x = 4\cos(\omega t)$

For x_c: $r^2 + r + 5 = 0 \Rightarrow$

$$r = \frac{-1 \pm \sqrt{1-20}}{2}$$

$$= \frac{-1 \pm i\sqrt{19}}{2}$$

$$x_c = \underbrace{c_1 e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{19}}{2}t\right) + c_2 e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{19}}{2}t\right)}_{\text{transient sol'n.}}$$

For the steady periodic part:

$$x = A\cos(\omega t) + B\sin(\omega t)$$
$$= C\cos(\omega t - \alpha)$$

where:

$$A = \frac{(k-m\omega^2)F_0}{(k-m\omega^2)^2 + (c\omega)^2}$$

$$B = \frac{c\omega F_0}{(k-m\omega^2)^2 + (c\omega)^2}$$

find from undet. coef. See end of notes for derivation.

$$C = \sqrt{A^2+B^2} = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}},$$

$\sin(\alpha) = \frac{B}{C}$

$\cos(\alpha) = \frac{A}{C}$

C is the amplitude: $m = 1$, $c = 1$, $k = 5$, $F = 4$

$$C(\omega) = \frac{4}{\sqrt{(5-\omega^2)^2 + \omega^2}}$$

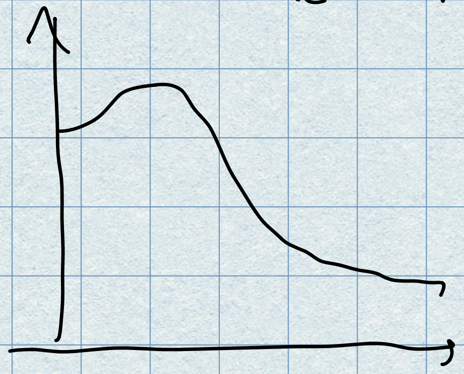
does it have critical pt?

$$C'(\omega) = \underbrace{4\left(-\tfrac{1}{2}\right)\left((5-\omega^2)^2+\omega^2\right)^{-\frac{3}{2}}}_{<0}\left[\underbrace{2(5-\omega^2)(-2\omega)+2\omega}\right]$$

$$\begin{aligned}&(10-2\omega^2)(-2\omega)+2\omega \\ &= (-10+2\omega^2+1)\cdot 2\omega \\ &= (2\omega^2-9)\cdot 2\omega\end{aligned}$$

Critical pts: $\omega = 0$, $\omega = \pm\frac{3}{\sqrt{2}}$

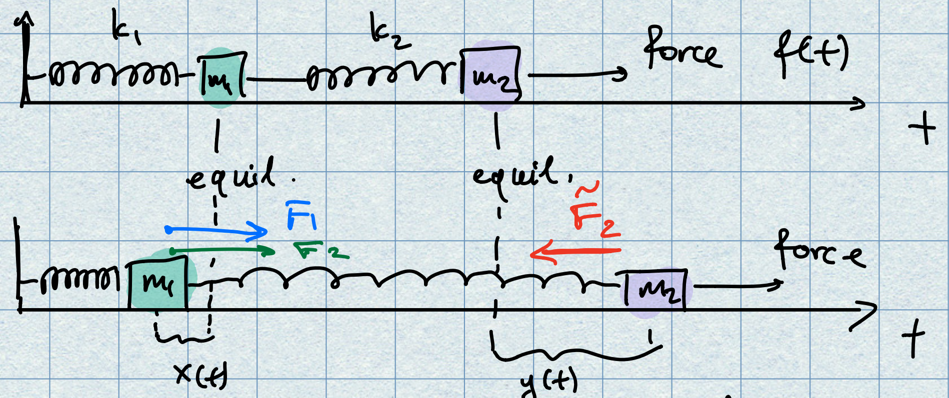
Practical resonance when $\omega = \frac{3}{\sqrt{2}}$



Exercise: Find α

4.1 Systems.

Prob. 30



$x(t) \rightarrow$ displ. of m_1 from equil. $\}$ signed
$y(t) \rightarrow$ " " m_2 " "

No friction

For m_1:
$$F_1 = -k_1 x$$
$$F_2 = \underbrace{+}_{+\text{ or }-?} k_2 \underbrace{(y - x)}_{\text{stretching of the spring}}$$

$$\Rightarrow m_1 x'' = -k_1 x + k_2 (y - x),$$

For m_2: force $f(t)$
$$\tilde{F}_2 = -k_2(y - x) = -F_2$$
$$m_2 y'' = -k_2(y - x) + f(t)$$

Motion of system
$$\begin{cases} m_1 x'' = -k_1 x + k_2(y - x) \\ m_2 y'' = -k_2(y - x) + f(t) \end{cases} \quad (*)$$

Seek pairs $(x(t), y(t))$ so that $(*)$ are satisfied simultaneously.

Reducing high order ODEs to 1st order systems.

CP 1: $u'' + u + \varepsilon u^3 = 0$ 2nd order eq'n

1st order system:

$$\begin{cases} u_1 = u \\ u_2 = u' = u_1' \end{cases}$$

$u_1' = u_2$

$u_2' = u_1'' = u'' = -u - \varepsilon u^3 = -u_1 - \varepsilon u_1^3$

1st order system

$$\begin{cases} u_1' = u_2 \\ u_2' = -u_1 - \varepsilon u_1^3 \end{cases}$$

Looking for $u(t) = u_1(t)$.

Higher order eq'n:

$$x^{(4)} + 6x''x' - 3x'x^2 + x = \cos(3t) \quad (*)$$

Turn into 1st order system.

as many var. as order.

$$\begin{cases} x_1 = x & (1) \\ x_2 = x' = x_1' & (2) \\ x_3 = x'' = x_1'' = x_2' & (3) \\ x_4 = x''' = x_1''' = x_2'' = x_3' & (4) \end{cases}$$

only 1st der.

$$x_4' = -6x''x' + 3x'x^2 - x + \cos(3t) \quad (5)$$

$$= -6x_3x_2 + 3x_2x_1^2 - x_1 + \cos(3t)$$

System: $(*)$

$$\begin{cases} x_1' = x_2 & (2) \\ x_2' = x_3 & (3) \\ x_3' = x_4 & (4) \\ x_4' = -6x_3x_2 + 3x_2x_1^2 - x_1 + \cos(3t) & (5) \end{cases}$$

How to find A, B: Undetermined Coef.

$$mx'' + cx' + kx = F_0 \cos(\omega t).$$

$$x_p = A\cos(\omega t) + B\sin(\omega t)$$

Plug in:

$$m\left(-A\omega^2 \cos(\omega t) - B\omega^2 \sin(\omega t)\right.$$

$$+ c\left(-A\omega \sin(\omega t) + B\omega \cos(\omega t)\right)$$

$$\left. + k\left(A\cos(\omega t) + B\sin(\omega t)\right)\right) = F_0 \cos(\omega t)$$

$$\begin{cases} (-A\omega^2 m + B\omega c + kA)\cos(\omega t) = F_0 \cos(\omega t) \\ (-B\omega^2 m - Ac\omega + kB)\sin(\omega t) = 0 \end{cases}$$

$$\left.\begin{array}{l} -mA\omega^2 + B\omega c + kA = F_0 \\ -B\omega^2 m - cA\omega + Bk = 0 \end{array}\right\} \text{2 eq's w/ 2 unknowns}$$

$$A(k - m\omega^2) + B\omega c = F_0$$

$$A(-c\omega) + B(k - \omega^2 m) = 0$$

$$A \cdot \cancel{(k - m\omega^2)}\, c\omega + B\omega^2 c^2 = F_0 c\omega$$

$$A\, \cancel{(-c\omega)}\, (k - m\omega^2) + B(k - m\omega^2)^2 = 0 \quad (+)$$

$$\Rightarrow \quad B = \frac{F_0 c\omega}{\omega^2 c^2 + (k - m\omega^2)^2}, \quad A = \frac{(k - \omega^2 m) \cdot F_0}{\omega^2 c + (k - m\omega^2)^2}$$

Find angle α: we have $m=1$, $c=1$, $k=5$, $F_0=9$

$$A = \frac{(k-m\omega^2)F_0}{(k-m\omega^2)^2 + (c\omega)^2}$$

$$B = \frac{c\omega F_0}{(k-m\omega^2)^2 + (c\omega)^2}$$

Angle will depend on ω. Take e.g. $\omega = \frac{3}{\sqrt{2}}$ (frequeny of practical resonance)

$$A = \frac{(5-\frac{9}{2})\cdot 9}{(5-\frac{9}{2})^2 + \frac{9}{2}} = \frac{2}{\frac{1}{4}+\frac{9}{2}} = \frac{8}{19}$$

$$B = \frac{\frac{3}{\sqrt{2}}\cdot 4}{(5-\frac{9}{2})^2 + \frac{9}{2}} = \frac{12}{\sqrt{2}}\cdot\frac{4}{19}$$

So: $C = \sqrt{A^2+B^2} = \frac{8}{\sqrt{19}}$

Want $\cos(\alpha) = \frac{A}{C} = \frac{1}{\sqrt{19}}$

$\sin(\alpha) = \frac{6}{\sqrt{2}}\cdot\frac{1}{\sqrt{19}} = \frac{3\sqrt{2}}{\sqrt{19}}$

so bec. $\cos(\alpha), \sin(\alpha) > 0$ α is in 1st quadrant & so

$$\alpha = \arccos\left(\frac{1}{\sqrt{19}}\right) = \arcsin\left(\frac{3\sqrt{2}}{\sqrt{19}}\right)$$

So for $\omega = \frac{3}{\sqrt{2}}$ the general sol'n becomes

$$x = c_1 e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{19}}{2} t\right) + c_2 e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{19}}{2} t\right)$$

$$+ \frac{8}{\sqrt{19}} \cos\left(\frac{3}{\sqrt{2}} t - \arccos\left(\frac{1}{\sqrt{19}}\right)\right)$$