Plan for today:
4.1-4.2

Learning goals:
1. Be able to turn high order systems into first order systems
2. Be able to solve simple constant coefficient systems by turning them into higher order equations that we can solve
3. Be able to use the method of elimination to solve systems

Announcements/Reminders
1. Read the textbook
2. Quiz 4 tomorrow.

Last time: Convert high order eqn into 1st order system.

Same for systems:

$$(*)\quad \left\{\begin{aligned} x'' &= (1-y)x \\ y'' &= (1-x)y \end{aligned}\right\} \text{ 2nd order system w/ 2 eq's.}$$

Turn into 1st order system.

Set $x_1 = x$

$x_2 = x_1' = x' \quad \Rightarrow \quad x_2' = x'' = (1-y_1)x_1$

$y_1 = y$

$y_2 = y' = y_1' \quad \Rightarrow \quad y_2' = y'' = (1-x_1)y_1$

1st order system w/ 4 eq's.
$$\left\{\begin{aligned} x_1' &= x_2 \\ x_2' &= (1-y_1)x_1 \\ y_1' &= y_2 \\ y_2' &= (1-x_1)y_1 \end{aligned}\right.$$

Fun exercise: solve $(*)$ w/ Matlab by converting into

Given small simple, can often solve by turning into high order eqn.

$$\begin{cases} x' = 8y & (1) \\ y' = -2x & (2) \end{cases}$$

$$(1) \Rightarrow x'' = 8y' \Rightarrow x'' = 8(-2x)$$

$$\Rightarrow x'' = -16x$$

$$\Rightarrow x'' + 16x = 0$$

$$r^2 + 16 = 0 \Rightarrow r = \pm 4i$$

$$x = c_1 \cos(4t) + c_2 \sin(4t)$$

To find y:

$$(1) \Rightarrow y = \frac{1}{8} x'$$

$$\Rightarrow y = \frac{1}{8}\left(-4c_1 \sin(4t) + 4c_2 \sin(4t)\right)$$

2 free parameters.

$$x = c_1 \cos(4t) + c_2 \sin(4t)$$

$$= C \cos(4t - \alpha)$$

$$C = \sqrt{c_1^2 + c_2^2}, \quad \left.\begin{aligned} \cos(\alpha) &= \frac{c_1}{C} \\ \sin(\alpha) &= \frac{c_2}{C} \end{aligned}\right\}$$

$$y = \frac{1}{8} x' = -\frac{1}{8} \cdot 4C \sin(4t - \alpha)$$

Sol'n to system: 2 free parameters.

$$(x, y) = \left(C \cos(4t - \alpha), -\frac{1}{2} C \sin(4t - \alpha)\right)$$

Can view ↑ as a curve in xy plane

depending on t.

What does the curve look like?

If $C \neq 0$

$$\cos(4t-\alpha) = \frac{x}{C}$$

$$\sin(4t-\alpha) = -\frac{2y}{C}$$

$$\cos^2(\theta) + \sin^2(\theta) = 1 \Rightarrow$$

$$\Rightarrow \quad \frac{x^2}{C^2} + \frac{4y^2}{C^2} = 1 \quad \rightarrow \text{ellipse.}$$

If $C = 0$: Just a point.

Method of Elimination

Const. coef system

$$\begin{cases} x'' - 3y' - 2x = 0 \\ y'' + 3x' - 2y = 0 \end{cases}$$

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Use differential op'tors. $\quad D = \frac{d}{dt}$

$$\begin{cases} D^2x - 3Dy - 2x = 0 \\ D^2y - 3Dx - 2y = 0 \end{cases}$$

$$\overbrace{(D^2-2)}^{L_1}x - \overbrace{3D}^{L_2}y = 0 \qquad L_1 x + L_2 y = f_1$$

$$\underbrace{3D}_{L_3}x + \underbrace{(D^2-2)}_{L_4}y = 0 \qquad L_3 x + L_4 y = f_2$$

$$\overbrace{3D}^{L_3}\,\overbrace{(D^2-2)}^{L_1}\,x - (3D)^2 y = 0 \quad \leftarrow \text{apply } L_3$$

$$\underbrace{(D^2-2)}_{L_1}\,\underbrace{3D}_{L_3}\,x + (D^2-2)^2 y = 0 \quad \leftarrow \text{apply } L_4$$

$\ominus$

bec. $L_1 L_3 = L_3 L_1$ for polyn. diff op'tors

$$\underbrace{\cancel{3D(D^2-2)x - (D^2-2)3Dx}}_{0} \quad \underbrace{-(3D)^2 y}\, - (D^2-2)^2 y = 0$$

$$(D^4 - 4D^2 + 4 + 9D^2)\,y = 0$$

$$\Rightarrow (D^4 + 5D^2 + 4)y = 0 \Rightarrow \underbrace{(D^2+1)(D^2+4)y = 0}_{\text{no } x!}$$

char. eq'n: $(r^2+1)(r^2+4)$

Roots $r = \pm i$, $r = \pm 2i$

$$y = C_1\cos(t) + C_2\sin(t) + C_3\cos(2t) + C_4\sin(2t)$$

To find x:

Option 1: (2) $\Rightarrow y'' + 3x' - 2y = 0$

$$\Rightarrow x' = \frac{1}{3}\left(-y'' + 2y\right)$$

Plug in y, integrate to find x.

Option 2: Do same as w/ y, eliminate y instead of x.

Apply L_4 to 1st line, L_2 to 2nd. (exercise)

Find: $x = a_1 \cos(t) + a_2 \sin(t) + a_3 \cos(2t) + a_4 \sin(2t)$

❗ 8 free parameters! $c_1, \dots, c_4$ for y
$a_1, \dots, a_4$ for x

Expect 4 free param. Use the system to remove 4.

$$x'' - 3y' - 2x = 0$$

$$\Rightarrow -a_1 \cos(t) - a_2 \sin(t) - 4a_3 \cos(2t) - 4a_4 \sin(2t)$$
$$-3\left(-c_1 \sin(t) + c_2 \cos(t) - 2c_3 \sin(2t) + 2c_4 \cos(2t)\right)$$
$$-2\left(a_1 \cos(t) + a_2 \sin(t) + a_3 \cos(2t) + a_4 \sin(2t)\right) = 0$$

$$-a_1 - 3c_2 - 2a_1 = 0 \Rightarrow c_2 = -a_1$$
$$-a_2 + 3c_1 - 2a_2 = 0 \Rightarrow c_1 = a_2$$
$$-4a_3 - 6c_4 - 2a_3 = 0 \Rightarrow c_4 = -a_3$$
$$-4a_4 + 6c_3 - 2a_4 = 0 \Rightarrow c_3 = a_4$$

Sol'n:

$$(x, y) = \big(a_1 \cos(t) + a_2 \sin(t) + a_3 \cos(2t) + a_4 \sin(2t),$$
$$a_2 \cos(t) - a_1 \sin(t) + a_4 \cos(2t) - a_3 \sin(2t)\big)$$

Gen. sol'n to system, 4 free param. //