Plan for today
§ 5.1

Learning goals/Important concepts:
1. Be able to rewrite a linear system in matrix form
2. Superposition principe
3. Structure of solutions to linear homogeneous systems
4. Check linear independence of solutions using the Wronskian

Announcements/Reminders
1. Quiz should be graded by Monday
2. Read the textbook
3. OH today 2-3
4. HW due tonight

Ch. 5.

Matrix valued functions

$$\underline{A}(t) = \underbrace{\begin{bmatrix} a_{11}(t) & a_{12}(t) & & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & & a_{2n}(t) \\ & & & \\ a_{m1}(t) & . & . \; . & a_{mn}(t) \end{bmatrix}}_{n \text{ columns}} \Big\} \, m \text{ rows}$$

Entries of $\underline{A}(t)$ are functions of t.

Differentiate:

$$\frac{d}{dt} \underline{A}(t) = \left[\frac{d}{dt} a_{ij} \right]$$

$$= \begin{bmatrix} \frac{d}{dt} a_{11}(t) & \cdot & \frac{d}{dt} a_{1n} \\ - - & \vdots & \\ \frac{d}{dt} a_{m1}(t) & \vdots & \frac{d}{dt} a_{mn} \end{bmatrix}$$

Ex:
$$\frac{d}{dt}\begin{bmatrix} 1 \\ 2t \\ \sin(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ \cos t \end{bmatrix}$$

If c scalar:
$$\frac{d}{dt}\left(c\,\underline{\underline{A}}(t)\right) = c\,\frac{d}{dt}\underline{\underline{A}}(t)$$

$$\frac{d}{dt}\Big(\underbrace{\underline{\underline{A}}(t)}_{m\times p}\ \underbrace{\underline{\underline{B}}(t)}_{p\times n}\Big) = \left(\frac{d}{dt}\underline{\underline{A}}(t)\right)\underline{\underline{B}}(t) + \underline{\underline{A}}(t)\left(\frac{d}{dt}\underline{\underline{B}}(t)\right)$$

compatible

$m\times p \quad p\times n$

matrix product.

Systems in Matrix Form

Given:

$$\begin{cases} x_1'(t) = p_{11}(t)\,x_1(t) + p_{12}(t)\,x_2(t) + \dots + p_{1n}(t)\,x_n(t) + f_1(t) \\ x_2'(t) = p_{21}(t)\,x_1(t) + p_{22}(t)\,x_2(t) + \dots + p_{2n}(t)\,x_n(t) + f_2(t) \\ \quad \vdots \\ x_n'(t) = p_{n1}(t)\,x_1(t) + p_{22}(t)\,x_2(t) + \dots + p_{nn}(t)\,x_n(t) + f_n(t) \end{cases}$$

n unknown fcts - n eq's.

$$\underline{\underline{X}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \qquad \underline{\underline{P}}(t) = \begin{bmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ \vdots & & \vdots \\ p_{n1}(t) & \cdots & p_{nn}(t) \end{bmatrix} \qquad \underline{\underline{f}}(t) = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

Rewrite:

$$\underline{\underline{x}}'(t) = \underline{\underline{P}}(t)\,\underline{\underline{x}}(t) + \underline{\underline{f}}(t) \qquad (*)$$

Ex: $x_1' = 2x_1 + x_2 + \sin(t)$

$x_2' = 3x_1 - 2x_2$

$$\underline{\underline{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \underline{\underline{P}} = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} \quad \underline{\underline{f}} = \begin{bmatrix} \sin(t) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \underbrace{\begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\text{matrix mult.}} + \begin{bmatrix} \sin(t) \\ 0 \end{bmatrix}$$

Existence & Uniqueness: If $\underline{\underline{P}}(t)$, $\underline{\underline{f}}(t)$ cont. on interval I containing number a, $\underline{\underline{b}}$ $n \times 1$ column vector then $(*)$ has unique soln on all of I satisfying $\underline{\underline{x}}(a) = \underline{\underline{b}}$.

Ex: $\begin{cases} x_1' = \sin(t)\,x_1 + 2x_3 + \ln(t+1) \\ x_2' = x_1 + e^t x_2 \\ x_3' = x_1 - x_2 + \cos(t) \end{cases}$

Step 1: Rewrite in matrix form.

$$\underline{\underline{x}}' = \underline{\underline{P}}(t)\,\underline{\underline{x}} + \underline{\underline{f}}(t) \qquad (*)$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \underline{\underline{P}}(t) = \begin{bmatrix} \sin(t) & 0 & 2 \\ 1 & e^t & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

no x_2 in 1st eq'n

3 x 3 matrix.

$$\underline{\underline{f}}(t) = \begin{bmatrix} \ln(t+1) \\ 0 \\ \cos(t) \end{bmatrix}$$

If $\underline{\underline{b}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ there is a unique sol'n to ⊛ w/ $\underline{\underline{x}}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ on $I = (-1, \infty)$

<u>System:</u> $x' = \underline{\underline{P}}(t)\, x$ homogeneous linear system

$x' = \underline{\underline{P}}(t)\, x + \underline{\underline{f}}(t)$ non-homog. linear system.

$\underline{\underline{f}} \neq \underline{\underline{0}}$

<u>Superposition Principle:</u>

homogeneous

If $\underline{\underline{x}}_1, \ldots, \underline{\underline{x}}_n$ are sol's to $\underline{\underline{x}}' = \underline{\underline{P}}(x)$ then

const. scalars

$$c_1 \underline{\underline{x}}_1(t) + \ldots + c_n \underline{\underline{x}}_n(t)$$

is also a sol'n.

"we can produce new sol's from known ones"

Q: What are the good building blocks?

Linear indepence

Def'n The sol's $\underline{\underline{x}}_1, \ldots, \underline{\underline{x}}_n$ of $\underline{\underline{x}}' = \overbrace{\underline{\underline{P}}(t)}^{n\times n}\underline{\underline{x}}$ are lin. independent if

$$c_1 \underline{\underline{x}}_1(t) + \ldots + c_n \underline{\underline{x}}_n(t) = 0 \Rightarrow c_1 = c_2 = \ldots = c_n = 0$$

constants.

Thm: If $\underline{\underline{x}}_1, \ldots, \underline{\underline{x}}_n$ are lin. indep. sols of $\underline{\underline{x}}' = \underset{n\times n}{\underline{\underline{P}}(t)}\underline{\underline{x}}$ then any sol'n of the system has form

$$\underline{x} = c_1 \underline{\underline{x}}_1(t) + \cdots + c_n \underline{\underline{x}}_n(t).$$

"Lin. indep. sols are good building blocks".

[Sol's of $\underline{\underline{x}}' = \underline{\underline{P}}(t)\underline{\underline{x}}$ form an n-dim'l vector space & $\underline{\underline{x}}_1, \ldots, \underline{\underline{x}}_n$ are a basis]

Check linear Indep.

Wronskian

Given: $\underline{\underline{x}}_1 = \begin{bmatrix} x_{11}(t) \\ \vdots \\ x_{n1}(t) \end{bmatrix}, \underline{\underline{x}}_2 = \begin{bmatrix} x_{12} \\ \vdots \\ x_{n2} \end{bmatrix}, \underline{\underline{x}}_n = \begin{bmatrix} x_{1n}(t) \\ \vdots \\ x_{nn}(t) \end{bmatrix}$

$$W(\underline{\underline{x}}_1(t), \ldots, \underline{\underline{x}}_n(t)) = \det\begin{bmatrix} x_{11}(t) & x_{12}(t) & & x_{1n}(t) \\ \vdots & \vdots & & \vdots \\ x_{n1}(t) & x_{n2}(t) & & x_{nn}(t) \end{bmatrix}$$

Note: we aren't taking derivatives of $\underline{\underline{x}}_j$

If $\underline{\underline{x}}_1, \ldots, \underline{\underline{x}}_n$ are sols of $\underline{\underline{x}}' = \underline{\underline{P}}(t)\underline{\underline{x}}$

then:

-> $\underline{\underline{x}}_1, \ldots, \underline{\underline{x}}_n$ lin. dependent on an interval I

$W(\underline{\underline{x}}_1, \ldots, \underline{\underline{x}}_n) \equiv 0$ on I

-> $\underline{\underline{x}}_1, \ldots, \underline{\underline{x}}_n$ lin. indep on I

=> $W(\underline{\underline{x}}_1, \ldots, \underline{\underline{x}}_n) \neq 0$ everywhere on I.

Ex: $\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$ $(*)$

$$\underline{\underline{A}} = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}$$

learn how to find them next week

Given sols: $\underline{\underline{x}}_1 = e^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} e^{3t} \\ -e^{3t} \end{bmatrix}$ $\underline{\underline{x}}_2 = e^{2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Find sol'n of $(*)$ w/ $\underline{\underline{x}}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ on $\mathbb{R}$.

Why is there a sol'n w/ $\underline{\underline{x}}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ on all of $\mathbb{R}^2$.

$\Rightarrow \underline{\underline{A}}$ cont. on $\mathbb{R}$ (const.).

Check lin. indep. of $\underline{\underline{x}}_1, \underline{\underline{x}}_2$

$$W(\underline{\underline{x}}_1, \underline{\underline{x}}_2) = \det \begin{bmatrix} e^{3t} & e^{2t} \\ -e^{3t} & -2e^{2t} \end{bmatrix}$$

$$= -2e^{5t} + e^{5t} = -e^{5t} \neq 0$$

on $\mathbb{R}$

$\Rightarrow \underline{\underline{x}}_1, \underline{\underline{x}}_2$ lin. indep.

$\overset{\text{thm}}{\Longrightarrow}$ $\underline{\text{any}}$ sol'n is $\underline{\underline{x}} = c_1 \underline{\underline{x}}_1(t) + c_2 \underline{\underline{x}}_2(t)$

__Want:__ $\underline{\underline{x}}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\Rightarrow c_1 \underline{\underline{x}}_1(0) + c_2 \underline{\underline{x}}_2(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} c_1 + c_2 = 1 \\ -c_1 - 2c_2 = 2 \end{cases} \Rightarrow \begin{cases} c_1 = 4 \\ c_2 = -3 \end{cases}$$

$$\underline{\underline{x}} = 4 \begin{bmatrix} e^{3t} \\ -e^{3t} \end{bmatrix} - 3 \begin{bmatrix} e^{2t} \\ -2e^{2t} \end{bmatrix}$$

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