Plan for today:

§ 5.2

Learning goals/important concepts:

1. Be able to solve a 1st order system using the eigenvalue method when the characteristic equation has distinct real roots.
2. Eigenvalue, eigenvector of a matrix. Characteristic equation

Reminders/announcements

1. Quiz grades will be posted today
2. Computer Project 2 due Friday
3. Read the textbook!

Discussed linear systems (method of elimination)

Today: Eigenvalue method.

$$\frac{d\underline{x}}{dt} = \underline{A}\,\underline{x} \quad (*), \quad \underline{A} \quad n \times n \text{ matrix}$$

const. coef.

Ex: $\dfrac{d\underline{x}}{dt} = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix} \underline{x}$ $\qquad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Want: n lin. indep. sols. for $(*)$ Any other sol'n will be linear comb. of them.

Exploration: $y'' + py' + q = 0$ $(*)$

const.

Set: $y = e^{\lambda t} \Rightarrow$ y is sol'n ex. when

$\lambda^2 + p\lambda + q = 0$

found λ, found sol'n $y = e^{\lambda t}$.

Write $(*)$ as system:

$$y = y_1, \quad y_2 = y' \implies \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \text{check!} \quad (\circledast)$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix}$$

Our educated guess $y = e^{\lambda t}$ becomes

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} e^{\lambda t} \\ \lambda e^{\lambda t} \end{bmatrix}.$$

So if $e^{\lambda t}$ solves $(*)$ then $e^{\lambda t}\underbrace{\begin{bmatrix} 1 \\ \lambda \end{bmatrix}}_{\text{const. vector dep. on } \lambda}$ solves $(\circledast)$

//.

Given $\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$, $\underline{\underline{A}}$ $n \times n$.

Try: $\underline{\underline{x}} = e^{\lambda t}\, \underbrace{\underline{\underline{v}}}_{\text{const. vector}}$ for some λ.

$$\left.\begin{aligned} \underline{\underline{x}}' &= \left(e^{\lambda t}\underline{\underline{v}}\right)' = \lambda e^{\lambda t}\underline{\underline{v}} \\ &\parallel \\ \underline{\underline{A}}\,\underline{\underline{x}} &= \underline{\underline{A}}\, e^{\lambda t}\underline{\underline{v}} \end{aligned}\right\}$$

$$\implies \underline{\underline{A}}\,\underline{\underline{v}} = \lambda \underline{\underline{v}} \iff \underline{\underline{A}}\,\underline{\underline{v}} = \lambda \underbrace{\underline{\underline{I}}}_{\text{identity matrix}} \underline{\underline{v}}$$

$$\iff \left(\underline{\underline{A}} - \lambda \underline{\underline{I}}\right)\underline{\underline{v}} = 0.$$

$$\underline{\underline{I}} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots \\ 0 & & & 1 \end{bmatrix}$$

<u>So:</u> if we can solve $(\underline{\underline{A}} - \lambda\underline{\underline{I}})\underline{\underline{v}} = \underline{\underline{0}}$ for <u>some</u> λ and <u>some</u> $\underline{\underline{v}} \neq \underline{\underline{0}}$

then $\quad e^{\lambda t} \underline{v}$ will solve $\underline{x}' = \underline{\underline{A}}\,\underline{x}$.

Fact

Can solve $(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{v} = \underline{0}$ for a $\underline{v} \neq \underline{0}$ exactly when $\underbrace{\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0}_{\text{characteristic eq'n of } \underline{\underline{A}}}$.

Def'n: A number λ (real, cplx, 0) is an eigenvalue of $\underline{\underline{A}}$ ($n \times n$) if

$$\underbrace{\det(\underline{\underline{A}} - \lambda \underline{\underline{I}})}_{\text{polynomial of deg. } n} = 0$$

An eigenvector associated w/ λ is a non-zero vector $\underline{v}$ such that

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{v} = \underline{0} \iff \underline{\underline{A}}\,\underline{v} = \lambda \underline{v}$$

matrix. scalar

special number, eigenvalue

Ex: $\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{\underline{\underline{A}}} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\underline{v}} = \underbrace{1}_{\lambda} \cdot \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\underline{v}}$

Method: $\underline{x}' = \underline{\underline{A}}\,\underline{x}$

1. Solve char. eq'n $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$.
   $\to$ n roots, n values for λ, $\lambda_1, \dots, \lambda_n$
2. For each λ_j find $\underline{v}_j$ eigenvector assoc. to λ_j

3. If process gives n lin. indep. $\underline{\underline{v}}_j$

Then $\quad x_1 = e^{\lambda_1 t} v_1, \ldots, x_n = e^{\lambda_n t} \underline{\underline{v}}_n$

are n lin. indep. sol's for $\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$

4. Any sol'n is

$$x = c_1 \underline{\underline{x}}_1 + \ldots + c_n \underline{\underline{x}}_n$$
$$= c_1 e^{\lambda_1 t} \underline{\underline{v}}_1 + \ldots + c_n e^{\lambda_n t} \underline{\underline{v}}_n$$

Fact 2

If λ_j are all distinct then method works.

Ex:

$$x_1' = 5x_1 - 6x_3$$
$$x_2' = 2x_1 - x_2 - 2x_3$$
$$x_3' = 4x_1 - 2x_2 - 4x_3.$$

$$\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}} \qquad \underline{A} = \begin{bmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix}$$

1. Solve char. eq'n.

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$$

$$\det\left(\begin{bmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix} - \overbrace{\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}}^{\lambda \underline{\underline{I}}} \right) = 0$$

$$\det\left(\begin{bmatrix} 5-\lambda & 0 & -6 \\ 2 & -1-\lambda & -2 \\ 4 & -2 & -4-\lambda \end{bmatrix} \right) = 0$$

$$(5-\lambda)\begin{vmatrix} -1-\lambda & -2 \\ -2 & -4-\lambda \end{vmatrix} - 6\begin{vmatrix} 2 & -1-\lambda \\ 4 & -2 \end{vmatrix} = 0$$

$$(5-\lambda)\big((1+\lambda)(4+\lambda)-4\big) - 6(\cancel{-4}+4\lambda\cancel{+4}) = 0$$

$$\cdots\cdots$$

$\lambda - \lambda^3 = 0.$ ← char. eq'n, polyn. in λ

Roots: $\lambda = 0, \pm 1.$ 3 distinct real roots,

⇒ Method works by Fact 2.

2. Find Eigenvectors, for each root.

→ $\lambda = 0.$

Looking for: $\underline{\underline{v}} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ so that

$$\underline{\underline{A}}\,\underline{\underline{v}} = \underbrace{0 \cdot \underline{\underline{v}}}_{\underline{\underline{0}}}$$

$$\begin{bmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{0\cdot\underline{\underline{v}}}$$

3rd line is const. multiple of 2nd.

① $5v_1 - 6v_3 = 0$

② $2v_1 - v_2 - 2v_3 = 0$

③ $\cancel{4v_1 - 2v_2 - 4v_3 = 0}$

① $\Rightarrow$ $v_1 = \frac{6}{5} v_3$

② $\Rightarrow$ $v_2 = 2v_1 - 2v_3 = 2\,\frac{6}{5} v_3 - 2v_3$

$= \frac{2}{5} v_3$

v_3 has no restrictions.

So for <u>any</u> v_3
$$\begin{bmatrix} \frac{6}{5} v_3 \\ \frac{2}{5} v_3 \\ v_3 \end{bmatrix}$$
is an eigenvector.

Can take $v_3 = 5$ $\Rightarrow$ $\begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix}$ eigenvector, so

$$c \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix} = c\, e^{0t} \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix}$$ is a sol'n of $\underline{x}' = \underline{\underline{A}}\, \underline{x}$

for any c.

Case $\lambda = 1$ will be discussed on Wednesday.