Plan for today

5.2, 5.5

A comment on RLC circuits

Learning goals:

1. Be able to solve a linear 1st order system for which the corresponding matrix has characteristic equation with complex roots or repeated roots using the eigenvalue method.

Announcements/Reminders

1. Solutions to Quiz 4 posted on Gradescope
2. Read the textbook!

Last time:

$$\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}} \qquad \underline{A} = \begin{bmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix}$$

Goal: 3 lin. indep. sols.

1. Solve char. eq'n. (to find eigenv.)

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0 \Rightarrow \lambda = 0, \pm 1.$$

2. For $\lambda = 0$ found eigenvector $\begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix}$,

so $x_1 = c\,\underset{\overset{\shortparallel}{1}}{e^{0t}} \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix}$ is a sol'n.

For $\lambda = 1$: find eigenvector.

Non-zero $\underline{v}$: $\underline{\underline{A}}\,\underline{\underline{v}} = 1 \cdot \underline{v} \iff (\underline{\underline{A}} - 1\underline{\underline{I}})\underline{v} = 0$

$$\Leftarrow \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} 4 & 0 & -6 \\ 2 & -2 & -2 \\ 4 & -2 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Elementary row operations: 1 interchange rows

2 multiplying row by non-zero scalar

3 add a row to another

$② \cdot \frac{1}{2} \to ②$, $② \leftrightarrow ①$

$$\begin{array}{c} ① \\ ② \\ ③ \end{array} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 4 & 0 & -6 & 0 \\ 4 & -2 & -5 & 0 \end{array}\right]$$

$② - 4 ① \to ②$

$③ - 4 \cdot ① \to ③$

$$\begin{array}{c} ① \\ ② \\ ③ \end{array} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 4 & -2 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right]$$

$③ - \frac{1}{2} ② \to ③$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_1 - v_2 - v_3 = 0 \Rightarrow v_1 = 3v_2$

$4v_2 - 2v_3 = 0 \Rightarrow v_3 = 2v_2$

So: $\begin{bmatrix} 3v_2 \\ v_2 \\ 2v_2 \end{bmatrix}$ is an eigenvector for any v_2

$$x_2 = C e^{t} \begin{bmatrix} 3v_2 \\ v_2 \\ 2v_2 \end{bmatrix}$$ is a sol'n for any v_2

Recall: if λ is an eigenvalue for $\underline{\underline{A}}$ then $e^{\lambda t}\underline{v}$ is a sol'n if $\underline{v}$ is an eigenv.

$$= C e^{t} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \quad \begin{pmatrix} \text{Choose} \\ v_2 = 1 \end{pmatrix}$$

For $\lambda = -1$: Exercise, find eigenvector then general sol'n:

$$x = C_1 \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix} + C_2 e^{t} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + C_3 e^{-t} \underline{\underline{V}}_3$$

eigenvector coming from $\lambda = -1$

lin. indep. bec. eigenvalues are distinct (not obvious that distinct eigenv. give n lin. indep. sols but true)

Case of complex eigenvalues.

$$\underline{\underline{x}}' = \underline{\underline{A}}\,x \qquad \underline{\underline{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \underline{\underline{A}} = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$$

Eigenvalues of $\underline{\underline{A}}$?

$$\det(\underline{\underline{A}} - \lambda I) = (3-\lambda)^2 + 16 = 0$$
$$\Rightarrow (3-\lambda)^2 = -16$$
$$\Rightarrow 3-\lambda = \pm i4$$
$$\Rightarrow \underbrace{\lambda = 3 \pm 4i}_{\text{conj. pair of roots.}}$$

$\underline{\lambda = 3+4i}$ Find eigenvector.

$$(A - (3+4i)I)\underline{\underline{v}} = \underline{\underline{0}}$$

$$\begin{bmatrix} -4i & -4 \\ 4 & -4i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-4iv_1 = 4v_2 \Rightarrow v_2 = -iv_1$ ← same

$4v_1 - 4iv_2 \Rightarrow v_1 = iv_2$

(so 2nd eqn doesn't add info)

$$\underline{\underline{v}} = \begin{bmatrix} iv_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix} v_2$$

So:

$\underline{\underline{x}}_1 = e^{(3+4i)t} \begin{bmatrix} i \\ 1 \end{bmatrix}$ is a sol'n.

Recall: looking for 2 lin. indep. sols.

Option 1: find eigenvector for $\lambda = 3-4i$, it will have complex entries.

$$\underline{\underline{x}}_2 = e^{(3-4i)t}\begin{bmatrix} a \\ b \end{bmatrix}, \quad a, b \text{ cplx.}$$

gen. sol'n $\quad c_1 \underline{\underline{x}}_1 + c_2 \underline{\underline{x}}_2$

cplx const.

Option 2: Observe:

if $\underline{x}' = \underline{\underline{A}}\,\underline{x}$, $\underline{\underline{A}}$ real entries

$$(\mathrm{Re}\,\underline{x})' = \underline{\underline{A}}\,(\mathrm{Re}\,\underline{x})$$

check the details!

If $\underline{x}$ solves $\underline{x}' = \underline{\underline{A}}\,\underline{x}$, then $\mathrm{Re}\,\underline{x}$, $\mathrm{Im}\,\underline{x}$ also do.

Take Re, Im of $e^{(3+4i)t}\begin{bmatrix} i \\ 1 \end{bmatrix}$.

$$e^{(3+4i)t}\begin{bmatrix} i \\ 1 \end{bmatrix} = \underbrace{e^{3t}(\cos(4t) + i\sin(4t))}_{\text{Euler's formula.}}\begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$= e^{3t}\begin{bmatrix} \cos(4t)\,i - \sin(4t) \\ \cos(4t) + i\sin(4t) \end{bmatrix}$$

$$= e^{3t}\begin{bmatrix} -\sin(4t) \\ \cos(4t) \end{bmatrix} + i\,e^{3t}\begin{bmatrix} \cos(4t) \\ \sin(4t) \end{bmatrix}$$

Real pt: $y_1 = e^{3t}\begin{bmatrix} -\sin(4t) \\ \cos(4t) \end{bmatrix}$

Im. pt: $y_2 = e^{3t}\begin{bmatrix} \cos(4t) \\ \sin(4t) \end{bmatrix}$

Found 2 sols!

Check that they are lin indep:

$$W(y_1, y_2) = \begin{vmatrix} e^{3t}(-\sin(4t)) & e^{3t}\cos(4t) \\ e^{3t}\cos(4t) & e^{3t}\sin(4t) \end{vmatrix}$$

$$= \ldots = -e^{6t} \neq 0 \quad \text{lin. indep.}$$

So gen. sol'n:

$$y = \underset{\uparrow}{c_1}\, e^{3t}\begin{bmatrix} -\sin(4t) \\ \cos(4t) \end{bmatrix} + \underset{\uparrow}{c_2}\, e^{3t}\begin{bmatrix} \cos(4t) \\ \sin(4t) \end{bmatrix}$$

real const.

Summary for cplx conj eigenv.:

→ Find eigenvalues

→ Find eigenvector $\underline{v}$ cor. to one of them

→ Find cplx valued sol'n (e.g. $e^{(3+4i)t}\begin{bmatrix} i \\ 1 \end{bmatrix}$)

→ Produce 2 lin. indep. sols by taking real & imaginary pts

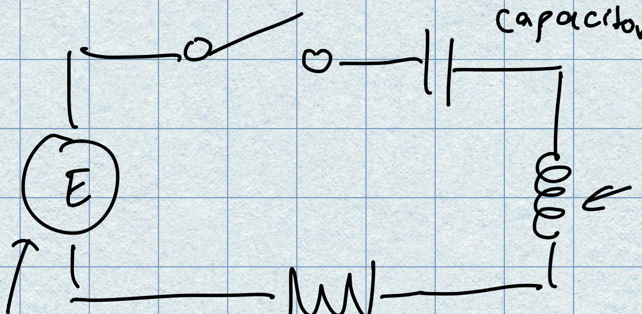
RLC circuits:

Mechanical systems mass-spring w/ damping & external force

$$mx'' + cx' + kx = f(t)$$

mass, damp., spring const, force

capacitor : C



inductor : L

voltage $E(t)$

resistor R ohms

same w/ different physical meanings

charge in capacitor $Q(t)$ satisfies

$$LQ'' + RQ' + \frac{1}{C}Q = E(t)$$

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t)$$

$I = \frac{dQ}{dt}$ current.

Can predict behavior of mass-spring by beh. of equivalent RLC circuit.