Plan for today:
5.5
Start 5.3

Learning goals:
1. Be able to solve a linear 1st order system for which the corresponding matrix has characteristic equation with repeated roots **of defect 1** using the eigenvalue method.
2. Be able to identify a phase plane portrait based on information about the eigenvalues of a system

Reminders/Announcements
1. No OH today - will be on piazza in the evening
2. Read the textbook!

Systems:

$\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$ $\quad \underline{\underline{A}}$ - const. coef. matrix $n\times n$

→ $\underline{\underline{A}}$ n distinct real eigenvalues (5.2)

→ $\underline{\underline{A}}$ complex conj. eigenvalues (5.2)

Today: $\underline{\underline{A}}$ repeated eigenv.

2×2 matrices

Ex 1: $\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$ $\quad \underline{\underline{A}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Seek: pair of lin. indep. sols.

Eigenv: $\lambda = 1$ repeated. (mult. 2)

Eigenv:

$(\underline{\underline{A}} - 1\cdot\underline{\underline{I}})\,\underline{\underline{v}} = \underline{\underline{0}}$, $\underline{\underline{v}} \neq 0$ $\iff$ $\underset{(2\times 2)}{\underline{\underline{0}}}\,\underline{\underline{v}} = \underset{(2\times 1)}{\underline{\underline{0}}}$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Any 2 lin. indep. vectors are eigenvectors!

eg. $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

lin. indep. vec. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ lin. indep.

$\rightarrow x_1 = e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\rightarrow x_2 = e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Gen sol'n: $x = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Had eigenv of multiplicity 2, could find 2 lin. indep. assoc. eigenv.

Ex 2: $\underline{x}' = \begin{bmatrix} 1 & -4 \\ 4 & 9 \end{bmatrix} \underline{x}$

$\underline{A}$

$\det(\underline{A} - \lambda \underline{I}) = 0 \Leftrightarrow \det \begin{bmatrix} 1-\lambda & -4 \\ 4 & 9-\lambda \end{bmatrix} = 0$

$(1-\lambda)(9-\lambda) + 16 = 0$

$9 - 10\lambda + \lambda^2 + 16 = 0$

$\lambda - 10\lambda + 25 = 0$

$\Rightarrow \lambda = 5$ repeated. mult. 2.

Eigenvectors:

$$(\underline{\underline{A}} - 5\underline{\underline{I}})\underline{\underline{v}} = \underline{\underline{0}}, \quad \underline{\underline{v}} \neq \underline{\underline{0}}$$

$$\begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

same! → $-4v_1 - 4v_2 = 0$

→ ~~$4v_1 + 4v_2 = 0$~~

Take $\underline{\underline{v}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

One sol'n: $\underline{\underline{x}}_1 = e^{5t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\underline{\underline{Q}}$ How do we find a second lin. indep. sol'n?

If eigenv. of mult. k has at most p assoc. lin. independent eigenvectors it is called defective. Defect = k − p

Ex 2: defect 2 − 1 = 1.

If k lin. indep. eigenv. then <u>complete</u>

<u>Ex 1</u>: 2 complete eigenvalue

Defective eigenvalues of defect 1:

$\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$, λ eigenvalue of defect 1

<u>Need</u>: good guess for format of sol'n.

Guess that sol'n has form

$\underline{\underline{x}}(t) = (\underline{\underline{v}}_1 t + \underline{\underline{v}}_2) e^{\lambda t}$

↑ what should these be?

$y'' - 2y' + y = 0$

↳ repeated root

$c_1 e^x + c_2 x e^x$

$$\frac{d}{dt} \underline{\underline{x}} = \underline{\underline{A}}\, \underline{\underline{x}}$$

$$\lambda e^{\lambda t} (\underline{\underline{v}}_1 t + \underline{\underline{v}}_2) + \underline{\underline{v}}_1 e^{\lambda t} = \underline{\underline{A}} (\underline{\underline{v}}_1 t + \underline{\underline{v}}_2) e^{\lambda t}$$

$$\lambda e^{\lambda t} t \underline{\underline{v}}_1 + \underline{\underline{v}}_2 \lambda e^{\lambda t} + \underline{\underline{v}}_1 e^{\lambda t} = \underline{\underline{A}} v_1 t e^{\lambda t} + \underline{\underline{A}} v_2 e^{\lambda t}$$

$$\underbrace{(\lambda \underline{\underline{v}}_2 + \underline{\underline{v}}_1 - \underline{\underline{A}} v_2)}_{\overset{?}{0}} e^{\lambda t} + \underbrace{(\lambda \underline{\underline{v}}_1 - \underline{\underline{A}}\, \underline{\underline{v}}_1)}_{\overset{?}{0}} t e^{\lambda t} = \underline{\underline{0}}$$

$$\Leftrightarrow \begin{cases} (\underline{\underline{A}} - \lambda \underline{\underline{I}}) \underline{\underline{v}}_1 = \underline{\underline{0}} & (1) \\ (\underline{\underline{A}} - \lambda \underline{\underline{I}}) \underline{\underline{v}}_2 = \underline{\underline{v}}_1 \neq 0 & (2) \end{cases} \quad (*)$$

If we can find $\underline{\underline{v}}_1, \underline{\underline{v}}_2 \neq 0$ satisfying $(*)$ then $\underline{\underline{x}} = (\underline{\underline{v}}_1 t + \underline{\underline{v}}_2) e^{\lambda t}$ will be a sol'n

Note:

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})^2 \underline{\underline{v}}_2 \overset{(2)}{=} (\underline{\underline{A}} - \lambda \underline{\underline{I}}) \underline{\underline{v}}_1 \overset{(1)}{=} \underline{\underline{0}}$$

Method: λ eigenvalue of defect 1.

Solve: $\begin{cases} (\underline{\underline{A}} - \lambda \underline{\underline{I}})^2 \underline{\underline{v}}_2 = \underline{\underline{0}} \\ (\underline{\underline{A}} - \lambda \underline{\underline{I}}) \underline{\underline{v}}_2 = \underline{\underline{v}}_1 \neq \underline{\underline{0}} \end{cases}$ ← start here

Then: $\underline{\underline{v}}_1 e^{\lambda t}$, $(\underline{\underline{v}}_1 t + v_2) e^{\lambda t}$ are lin. indep. sol's.

Note: v_1 is an eigenvector

v_2 is **not** an eigenvector (generalized eigenvector)

Back to example:

Ex 2: $\underline{\underline{x}}' = \begin{bmatrix} 1 & -4 \\ 4 & 9 \end{bmatrix} \underline{\underline{x}}$ (matrix $\underline{\underline{A}}$) 5 eigenvalue, defect 1

$$(\underline{\underline{A}} - 5\underline{\underline{I}}) = \begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix}$$

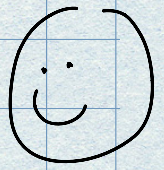
Looking for $\underline{\underline{v}}_2 \neq 0$ so that

$$\begin{cases} \begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix}^2 \underline{\underline{v}}_2 = \underline{\underline{0}} & ① \\ \begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix} \underline{\underline{v}}_2 = \underline{\underline{v}}_1 \neq \underline{\underline{0}} & ② \end{cases}$$

$$\begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix}^2 = \begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So: (1) gives no restriction

Must satisfy (2). Try a $\underline{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

then $\begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ 8 \end{bmatrix} \neq 0$ 

Any $\underline{v}_2$ which is not an eigenvector would work. Can't take $\underline{v}_2 = c \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

↑ eigenvector we found before.

Take: $\underline{v}_1 = \begin{bmatrix} -8 \\ 8 \end{bmatrix}, \quad \underline{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Pair of lin. indep. sols:

$$\underline{x}_1 = e^{5t} \underbrace{\begin{bmatrix} -8 \\ 8 \end{bmatrix}}_{\underline{v}_1}, \quad \underline{x}_2 = e^{5t} \left(\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\underline{v}_2} + \underbrace{\begin{bmatrix} -8 \\ 8 \end{bmatrix}}_{\underline{v}_1} t \right)$$

building blocks.

Gen. sol'n:

$$x(t) = c_1 e^{5t} \begin{bmatrix} -8 \\ 8 \end{bmatrix} + c_2 e^{5t} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -8 \\ 8 \end{bmatrix} t \right)$$