Plan for today:

5.3

Learning Goals/Important Concepts:

1. Be able to match a phase portrait to a linear system given its eigenvalues and/or general solution and vice versa
2. Proper/Improper nodal source/sink
3. Saddle point
4. Spiral sink/source
5. Be familiar with the pictures on pages 316-317

Reminders

1. Read the textbook!

$$\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$$

distinct eigenv., repeated, complex eigenv.

Phase Plane Portraits

For $\underline{\underline{A}}$ 2x2

$$\underline{\underline{x}} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

View $(x(t), y(t))$ as a curve on xy plane

Draw velocity vectors of those curves

Relate eigenvalues w/ graphs of velocity vectors.

Note: $(x'(t), y'(t))$ is known when we are given

$$\underline{\underline{x}}' = \underline{\underline{A}}\, x.$$

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$$

pplane 8

1. Distinct real e.v. of opposite signs.

$$x' = -\frac{5}{7}x + \frac{6}{7}y$$

$$y' = \frac{18}{7}x - \frac{2}{7}y \qquad \text{e.v. } 1, -2$$

Sol'n:

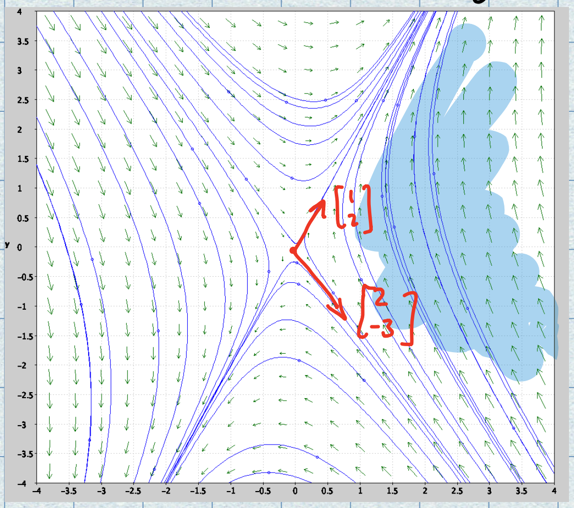
$$\underline{x}(t) = c_1 e^{t}\begin{bmatrix}1\\2\end{bmatrix} + c_2 e^{-2t}\begin{bmatrix}2\\-3\end{bmatrix}$$

$t\to\infty$: $\longrightarrow \infty$; $t\to\infty$: $\longrightarrow 0$

How does curve behave as $t\to\infty$?

As $t\to\infty$ we get "almost" a multiple of $\begin{bmatrix}1\\2\end{bmatrix}$

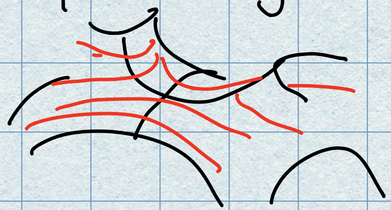
At $t\to-\infty$ we get almost a multiple of $\begin{bmatrix}2\\-3\end{bmatrix}$



$c_1>0$

$c_2>0$

origin is a saddle pt for the system.



2. Distinct real e.v., both negative

$$x' = -\frac{25}{7}x + \frac{2}{7}y$$

$$y' = \frac{6}{7}x - \frac{27}{7}y \qquad \text{e.v. } -3, -4$$

Sol'n:

$$\underline{x}(t) = c_1 e^{-3t}\begin{bmatrix}1\\2\end{bmatrix} + c_2 e^{-4t}\begin{bmatrix}2\\-3\end{bmatrix}$$

What happens to velocities?

$$\underline{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$\underline{x}'(t) = -3c_1 e^{-3t}\begin{bmatrix}1\\2\end{bmatrix} - 4c_2 e^{-4t}\begin{bmatrix}2\\-3\end{bmatrix}$$

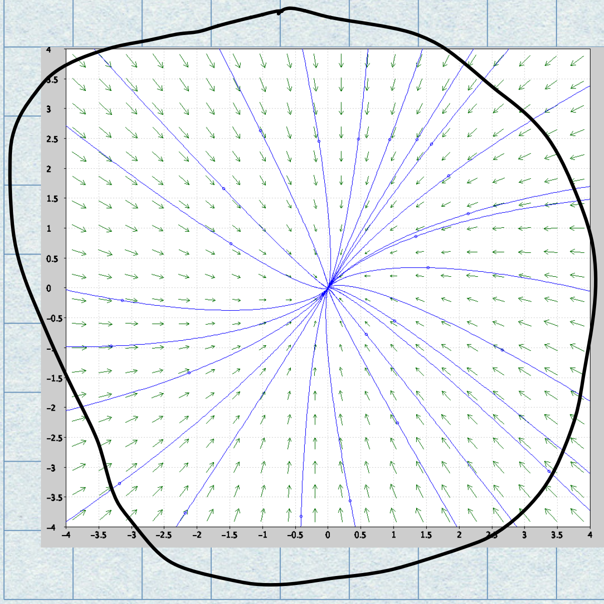
$$= e^{-3t}\left(-3c_1\begin{bmatrix}1\\2\end{bmatrix} \underbrace{-4c_2 e^{-t}\begin{bmatrix}2\\-3\end{bmatrix}}_{\xrightarrow{t\to\infty} 0}\right)$$

As $t \to \infty$ velocity becomes almost tang. to $-3c_1\begin{bmatrix}1\\2\end{bmatrix}$

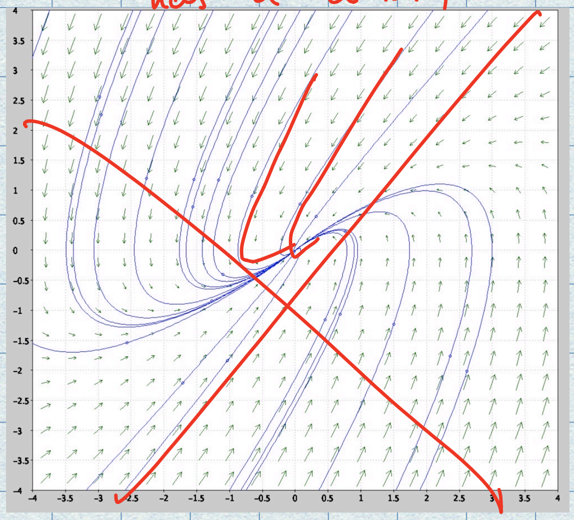
Also from (*) $\underline{x}(t) \to \underline{0}$ as $t \to \infty$

How to tell between H & D?

Once c_1 is fixed, $e^{-3t}\left(-3c_1\begin{bmatrix}1\\2\end{bmatrix}\right)$ points in the same direction for all t, so for large t $\underline{x}(t)$ should not change direction.



has a U-turn



3. Distinct real e.v., both positive.

$$x' = \frac{25}{7}x - \frac{2}{7}y$$
$$y' = -\frac{6}{7}x + \frac{27}{7}y$$

} (**)

Sol'n:

$$\underline{x}(t) = c_1 e^{3t}\begin{bmatrix}1\\2\end{bmatrix} + c_2 e^{4t}\begin{bmatrix}2\\-3\end{bmatrix}$$

Time Reversal

If $\underline{x}(t)$ solves $\underline{x}'(t) = \underline{A}\,\underline{x}(t)$

↳ const. coef.

then $\tilde{\underline{x}}(t) = \underline{x}(-t)$ satisfies

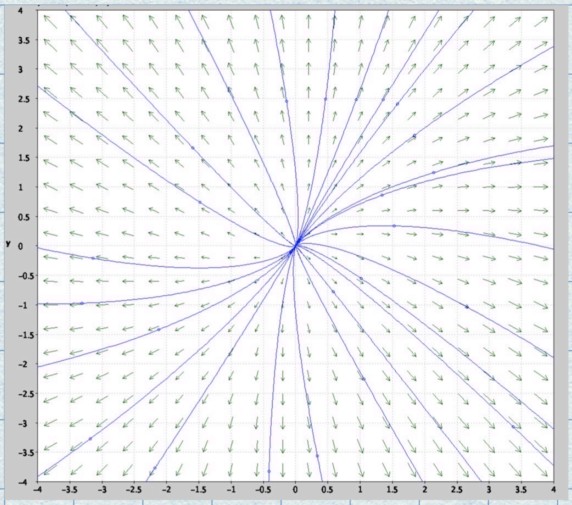
$$\tilde{\underline{x}}'(t) = -\underline{x}'(-t) = -\underline{A}\,\underline{x}(-t)$$
$$= -\underline{A}\,\tilde{\underline{x}}(t)$$

So: $\tilde{\underline{x}}(t)$ solves $\tilde{\underline{x}}' = -\underline{A}\,\underline{x}$

If λ eigenv. of $\underline{A}$ then $-\lambda$ is an eigenv. of $-\underline{A}$.

(**) same as (**) w/ matrix of opposite sign.

Sol'n: same as for (**) but with reversed time: phase portrait same w/ velocities pointing other way.



Terminology:

The origin is a <u>node</u> if

1. Either every traj approaches 0 as $t \to \infty$ <u>or</u> every traj. recedes away from 0

AND

2. Every traj. is tang. to a straight line through the origin at the origin.

If every traj. $\to 0$ as $t \to \infty$ then origin is a <u>sink</u>.

If every traj recedes from origin then origin is a <u>source</u>.

See pictures at the bottom!

Up to here on Monday

4. Distinct real, one 0 one negative

$$x' = -6x + 6y$$
$$y' = 9x + 9y$$

Sol'n:

$$\underline{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

5. Distinct real, one 0 one positive.

$$x' = 6x - 6y$$
$$y' = -9x - 9y$$

6. Complex, purly imaginary

$$x' = -4y$$
$$y' = x$$

Sol'n:

$$x(t) = -2c_1 \sin(2t) + 2c_2 \cos(2t)$$
$$y(t) = c_1 \cos(2t) + c_2 \sin(2t)$$

7. complex, real part < 0.

$$x' = -3x + 4y$$
$$y' = -4x - 3y$$

$\lambda = -3 \pm i4$

$$\underline{\underline{x}}(t) = a e^{-3t} \begin{bmatrix} \cos(4t) \\ \sin(4t) \end{bmatrix} + b e^{-3t} \begin{bmatrix} \sin(4t) \\ -\cos(4t) \end{bmatrix}$$

8. complex, real pt > 0.

$$x' = 3x - 4y$$
$$y' = 4x + 3y$$

$\lambda = +3 \pm i4$

$$\underline{\underline{x}}(t) = a e^{3t} \begin{bmatrix} \cos(4t) \\ \sin(4t) \end{bmatrix} + b e^{3t} \begin{bmatrix} \sin(4t) \\ -\cos(4t) \end{bmatrix}$$

9. Repeated, defect 0, negative

$x' = -x$ $\quad x = a e^{-t}$

$y' = -y$ $\quad y = b e^{-t}$

10. Repeated, defect 0, positive

$$x' = x$$
$$y' = y$$

$$x = a e^{t}$$
$$y = b e^{t}$$

11 Repeated, defect 1, negative.

$$x' = -y$$
$$y' = x - 2y$$

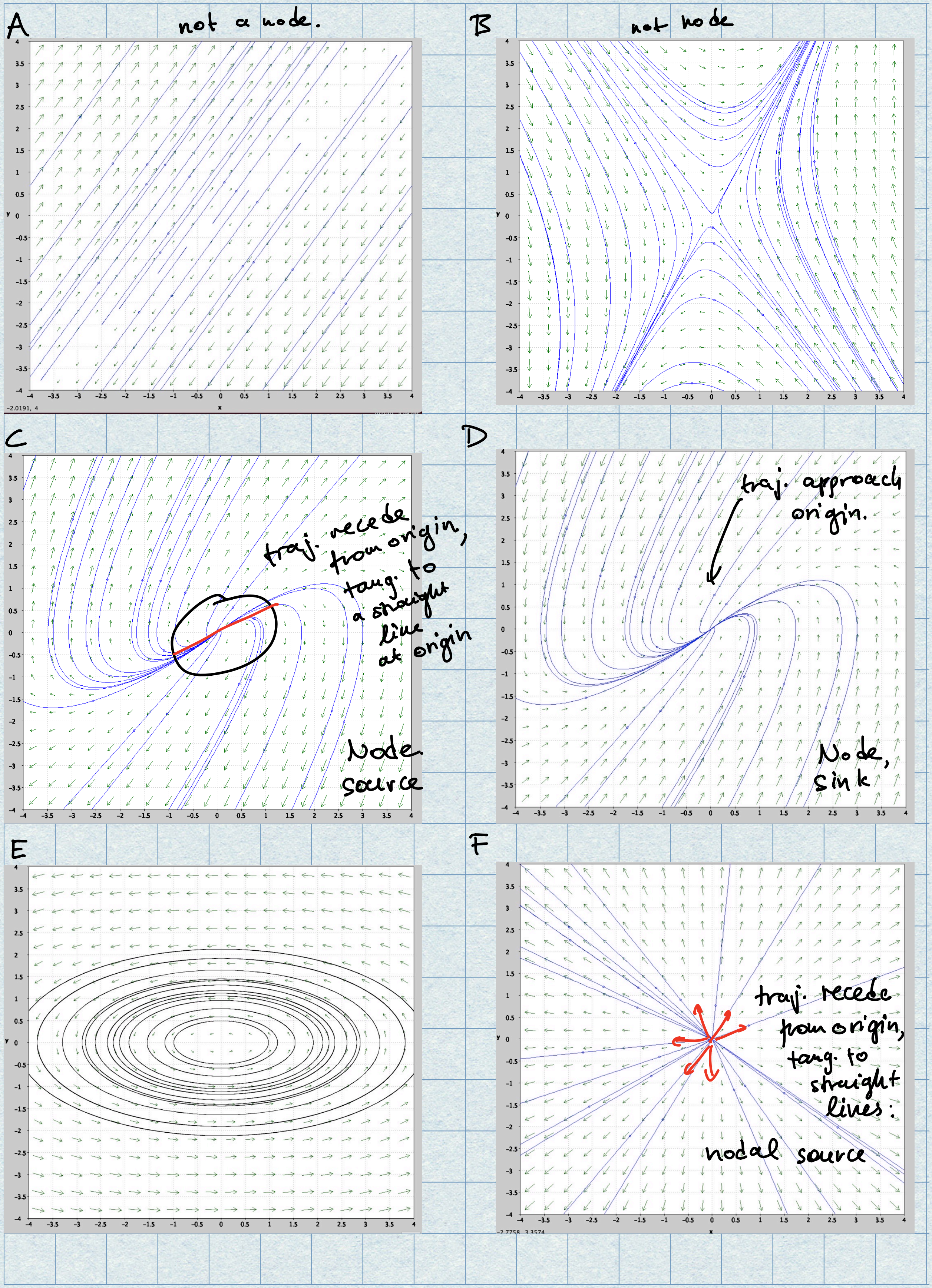
Sol'n:

$$\underline{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} (-t) + \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) e^{-t}$$

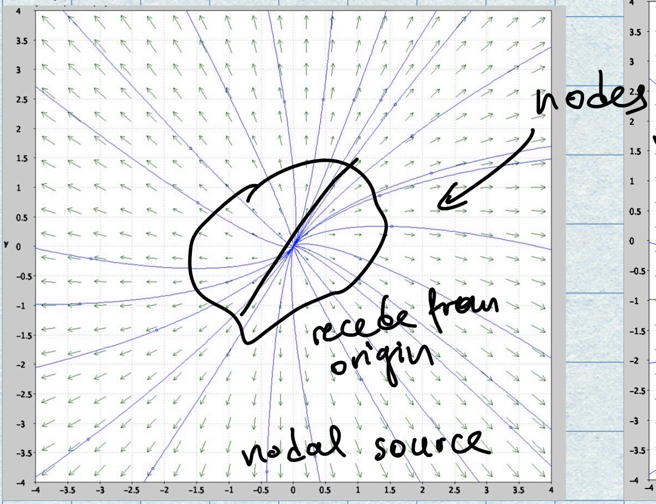
12. Repeated, defect 1, positive.

$$x' = y$$
$$y' = -x + 2y$$

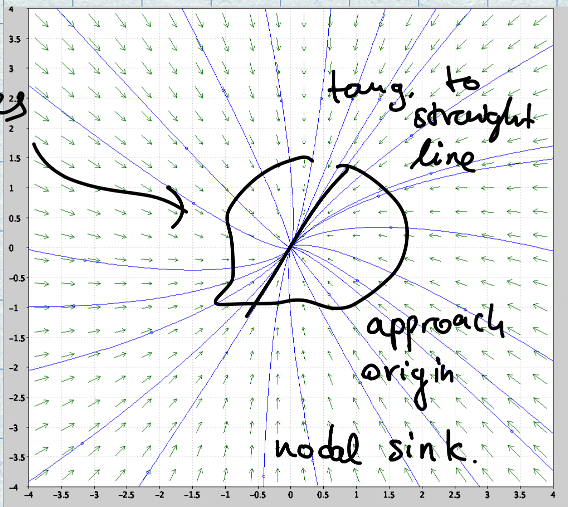
$$\underline{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) e^{t}$$


A
not a node.
B
not node
C
traj. recede from origin, tang. to a straight line at origin
Node source
D
traj. approach origin.
Node, sink
E
F
traj. recede from origin, tang. to straight lines:
nodal source

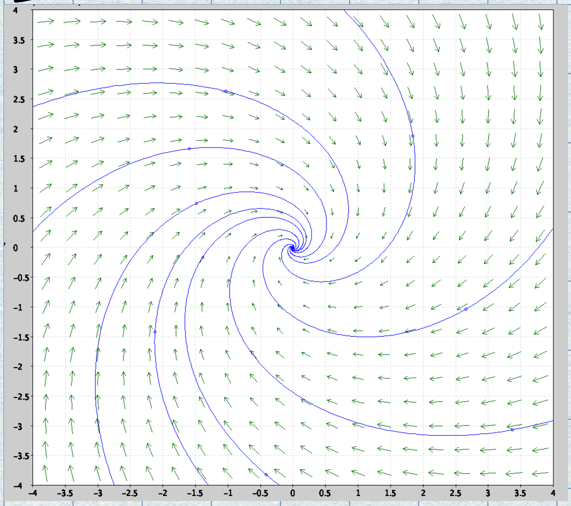
G


nodes
recede from origin
nodal source

H


tang. to straight line
approach origin
nodal sink.

I



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