Plan for today:

5.3

Learning Goals/Important Concepts:

1. Be able to match a phase portrait to a linear system given its eigenvalues and/or general solution and vice versa
2. Proper/Improper nodal source/sink
3. Saddle point
4. Spiral sink/source
5. Be familiar with the pictures on pages 316-317 → eigenvalues w/ pictures

Reminders

1. Read the textbook!

$$\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}$$

distinct eigenv., repeated, complex eigenv.

Phase Plane Portraits

For $\underline{\underline{A}}$ 2×2

$$\underline{\underline{x}} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

View $(x(t), y(t))$ as a curve on xy plane

Draw velocity vectors of those curves

Relate eigenvalues w/ graphs of velocity vectors.

Note: $(x'(t), y'(t))$ is known when we are given

$$\underbrace{\underline{\underline{x}}'}_{\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}} = \underline{\underline{A}}\,x.$$

pplane 8

1. Distinct real e.v. of opposite signs.

$$x' = -\frac{5}{7}x + \frac{6}{7}y$$

$$y' = \frac{18}{7}x - \frac{2}{7}y \qquad \text{e.v. } 1, -2$$

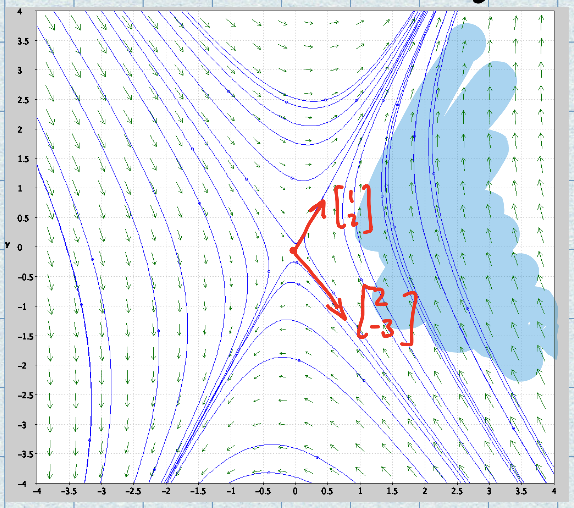
Sol'n:

$$\underline{x}(t) = c_1 \underbrace{e^{t}}_{t\to\infty \;\longrightarrow\; \infty} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \underbrace{e^{-2t}}_{t\to\infty \;\longrightarrow\; 0} \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

How does curve behave as $t \to \infty$?

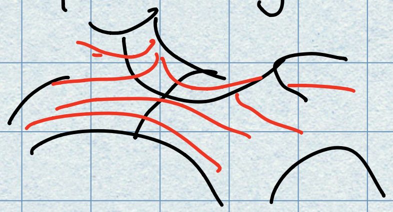
As $t \to \infty$ we get "almost" a multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

At $t \to -\infty$ we get almost a multiple of $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$



$c_1 > 0$
$c_2 > 0$

Origin is a saddle pt for the system.



2. Distinct real e.v., both negative

$$x' = -\frac{25}{7}x + \frac{2}{7}y$$

$$y' = \frac{6}{7}x - \frac{27}{7}y \qquad \text{e.v. } -3, -4$$

Sol'n:

$$\underline{x}(t) = c_1 e^{-3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-4t} \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

What happens to velocities?

$$\underline{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$\underline{x}'(t) = -3c_1 e^{-3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 4c_2 e^{-4t}\begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

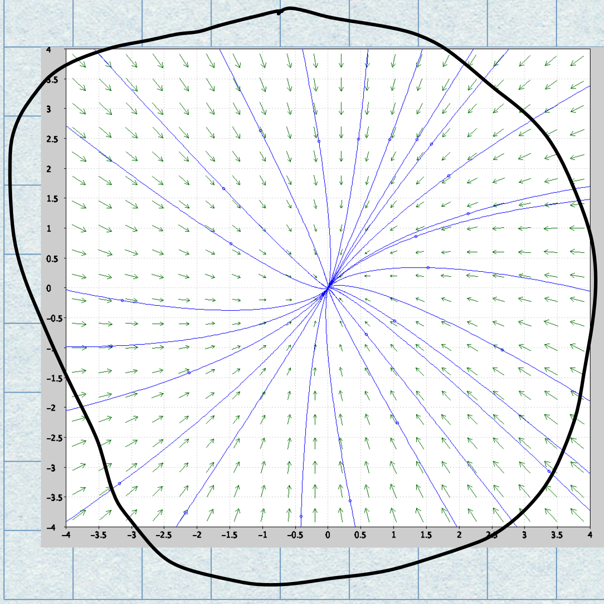
$$= e^{-3t}\left(\underbrace{-3c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix}} \; \underbrace{-4c_2 e^{-t}\begin{bmatrix} 2 \\ -3 \end{bmatrix}}_{\xrightarrow{t\to\infty} 0} \right)$$

As $t \to \infty$ velocity becomes almost tang. to $-3c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

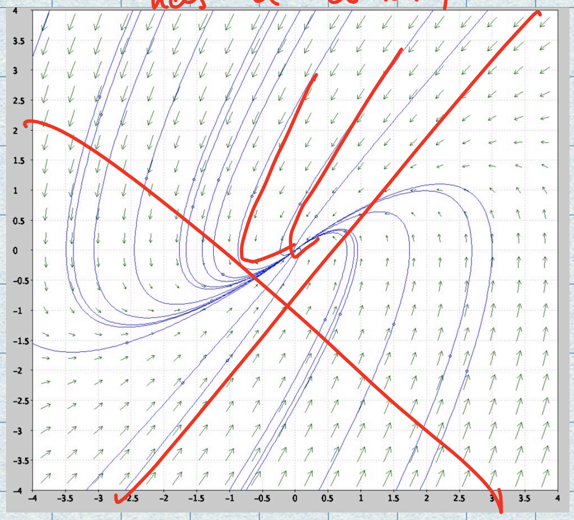
Also from $\ast$ $\underline{x}(t) \to \underline{0}$ as $t \to \infty$

How to tell between H & D?

Once c_1 is fixed, $e^{-3t}\left(-3c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ points in the same direction for all t, so for large t $\underline{x}(t)$ should not change direction.



has a U-turn



3. Distinct real e.v., both positive.

$$x' = \frac{25}{7}x - \frac{2}{7}y$$
$$y' = -\frac{6}{7}x + \frac{27}{7}y$$

(**)

Sol'n:

$$\underline{x}(t) = c_1 e^{3t}\begin{bmatrix}1\\2\end{bmatrix} + c_2 e^{4t}\begin{bmatrix}2\\-3\end{bmatrix}$$

Time Reversal

If $\underline{x}(t)$ solves $\underline{x}'(t) = \underline{A}\,\underline{x}(t)$

const. coef.

then $\tilde{\underline{x}}(t) = \underline{x}(-t)$ satisfies

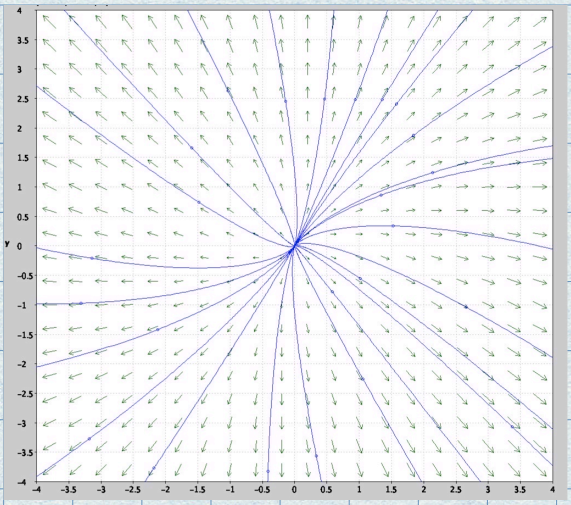
$$\tilde{\underline{x}}'(t) = -\underline{x}'(-t) = -\underline{A}\,\underline{x}(-t)$$
$$= -\underline{A}\,\tilde{\underline{x}}(t)$$

So: $\tilde{\underline{x}}(t)$ solves $\tilde{\underline{x}}' = -\underline{A}\,\underline{x}$

If λ eigenv. of $\underline{A}$ then $-\lambda$ is an eigenv. of $-\underline{A}$.

(**) same as (**) w/ matrix of opposite sign.

Sol'n: same as for (**) but with reversed time: phase portrait same w/ velocities pointing other way.



Terminology:

The origin is a node if

1. Either every traj approaches 0 as $t \to \infty$ or every traj. recedes away from 0

AND

2. Every traj. is tang. to a straight line through the origin at the origin.

If every traj. $\to 0$ as $t \to \infty$ then origin is a sink.

If every traj recedes from origin then origin is a source.

See pictures at the bottom!

Up to here on Monday

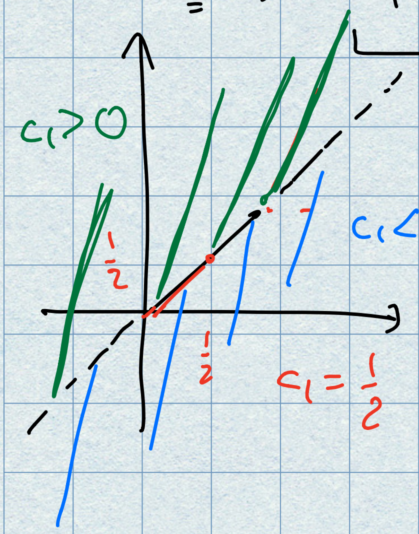
4. Distinct real, one 0 one negative

$$x' = -6x + 6y$$
$$y' = 9x + 9y$$

eigenv. 0, -3

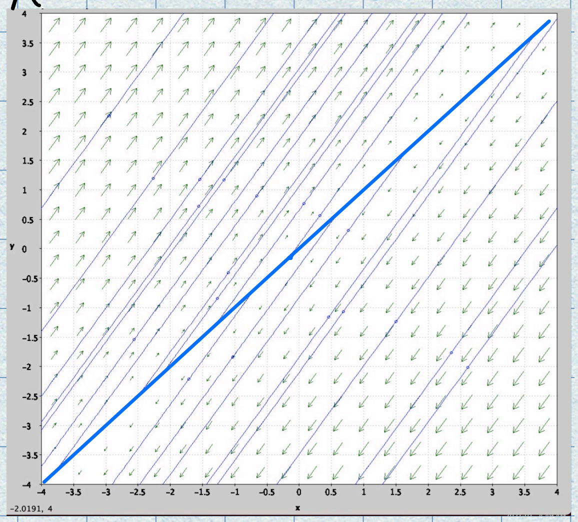
Sol'n:

$$\underline{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$



$c_1 = 1$

$c_2 = 1$



5. Distinct real, one 0 one positive.

$$x' = 6x - 6y$$
$$y' = -9x - 9y$$

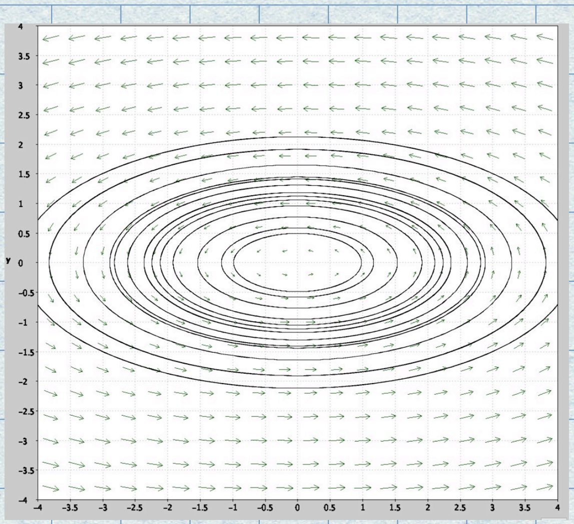
6. Complex, purely imaginary

$$x' = -4y$$
$$y' = x$$

Sol'n:

$$x(t) = -2c_1 \sin(2t) + 2c_2 \cos(2t)$$
$$y(t) = c_1 \cos(2t) + c_2 \sin(2t)$$

→ ellipses!



7. complex, real part < 0.

$$x' = -3x + 4y \qquad \lambda = -3 \pm i4$$
$$y' = -4x - 3y$$

$$\underline{x}(t) = a e^{-3t} \begin{bmatrix} \cos(4t) \\ \sin(4t) \end{bmatrix} + b e^{-3t} \begin{bmatrix} \sin(4t) \\ -\cos(4t) \end{bmatrix}$$

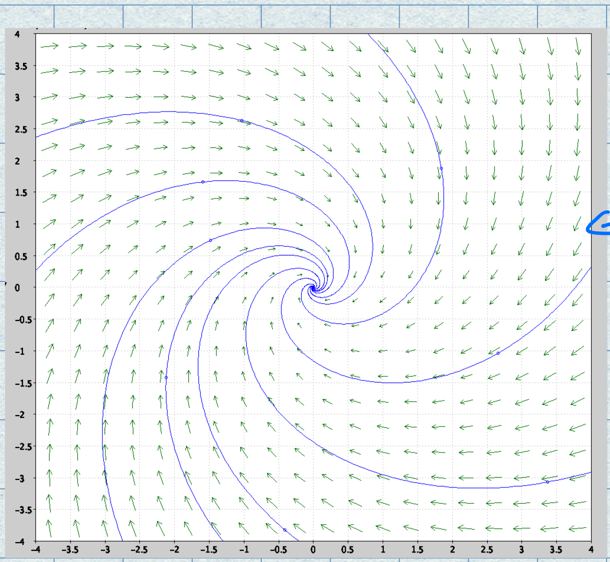
$$\left(x(t) = a e^{-3t} \cos(4t) + b e^{-3t} \sin(4t) \right)^2$$
$$\left(y(t) = a e^{-3t} \sin(4t) - b e^{-3t} \cos(4t) \right)^2$$

$$x^2(t) = a^2 e^{-6t} \cos^2(4t) + \cancel{2abe^{-6t} \cos(4t)\sin(4t)} + b^2 e^{-6t} \sin^2(4t)$$

$$y^2(t) = a^2 e^{-6t} \sin^2(4t) - \cancel{2abe^{-6t} \cos(4t)\sin(4t)} + b^2 e^{-6t} \cos^2(4t) \quad +$$

$$x^2(t) + y^2(t) = (a^2 + b^2) e^{-6t} \leftarrow$$

can think of as circle of reducing radius as $t \to \infty$

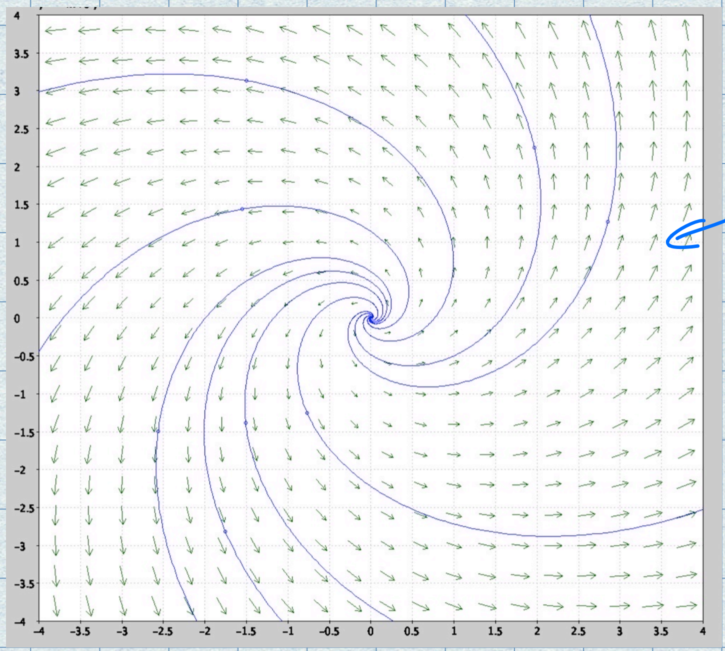


spiral sink

8. complex, real pt > 0.

$$x' = 3x - 4y \qquad \lambda = +3 \pm i4$$
$$y' = 4x + 3y$$

$$\underline{x}(t) = a e^{3t} \begin{bmatrix} \cos(4t) \\ \sin(4t) \end{bmatrix} + b e^{3t} \begin{bmatrix} \sin(4t) \\ -\cos(4t) \end{bmatrix}$$



spiral source

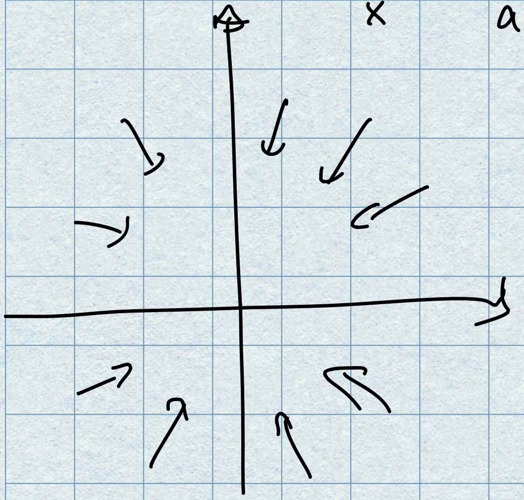
9. Repeated, defect 0; negative

$x' = -x \qquad x = a e^{-t}$

$y' = -y \qquad y = b e^{-t}$

If $a \neq 0$

$\frac{y}{x} = \frac{b}{a}$ ← straight line through origin,



← proper nodal sink

10. Repeated, defect 0, positive

$x' = x$

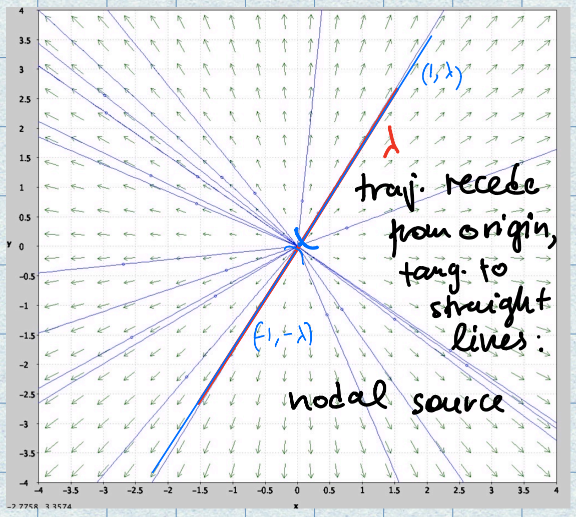
$y' = y$

$x = a e^{t}$

$y = b e^{t}$

at most one pair of curves tang. to same line.

proper nodal source.



Want slope λ at origin

$\begin{cases} y = -\lambda e^{t} \\ x = -e^{t} \end{cases} \qquad \begin{cases} y = \lambda e^{t} \\ x = e^{t} \end{cases}$

11. Repeated, defect 1, positive.

$$x' = y$$
$$y' = -x + 2y$$

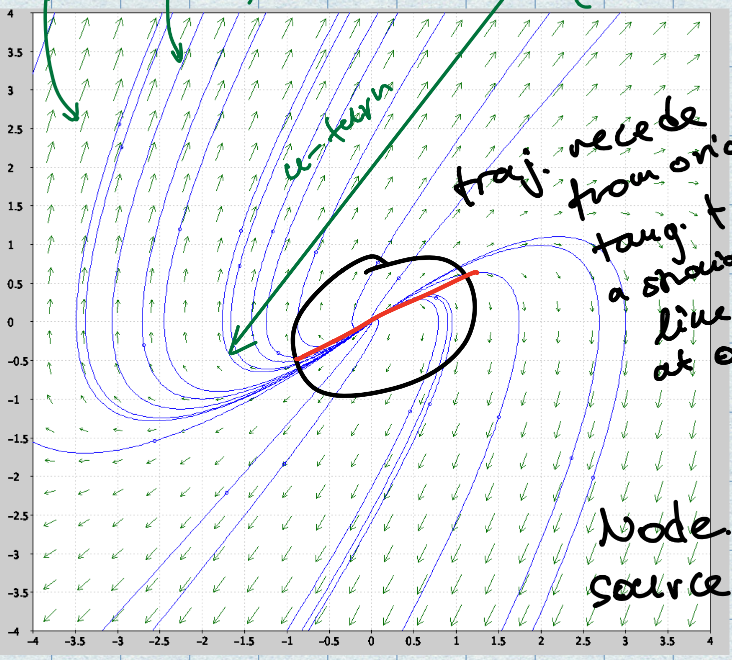
$$\underline{x}(t) = c_1 \begin{bmatrix}1\\1\end{bmatrix} e^{t} + c_2 \left(\begin{bmatrix}1\\1\end{bmatrix} t + \begin{bmatrix}3\\2\end{bmatrix} \right) e^{t} \xrightarrow{t \to -\infty} 0$$

$$\underline{x}'(t) = c_1 \begin{bmatrix}1\\1\end{bmatrix} e^{t} + c_2 \begin{bmatrix}1\\1\end{bmatrix} e^{t} + c_2 \left(\begin{bmatrix}1\\1\end{bmatrix} t + \begin{bmatrix}3\\2\end{bmatrix} \right) e^{t}$$

$$= te^{t} \left(c_2 \begin{bmatrix}1\\1\end{bmatrix} + \underbrace{\frac{1}{t} \left(c_1 \begin{bmatrix}1\\1\end{bmatrix} + c_2 \begin{bmatrix}1\\1\end{bmatrix} + c_2 \begin{bmatrix}3\\2\end{bmatrix} \right)}_{t \to \pm\infty \;\longrightarrow\; 0} \right)$$

when $t \to \pm\infty$ expect sth roughly // to $\begin{bmatrix}1\\1\end{bmatrix}$ $(c_2 \neq 0)$. If $\begin{cases} t \to +\infty & \text{pos. multiple of } c_1 \begin{bmatrix}1\\1\end{bmatrix} \\ t \to -\infty & \text{neg. mult. of } c_1 \begin{bmatrix}1\\1\end{bmatrix} \end{cases}$

$x'(t)$ $x'(t)$



improper nodal source

12 Repeated, defect 1, negative.

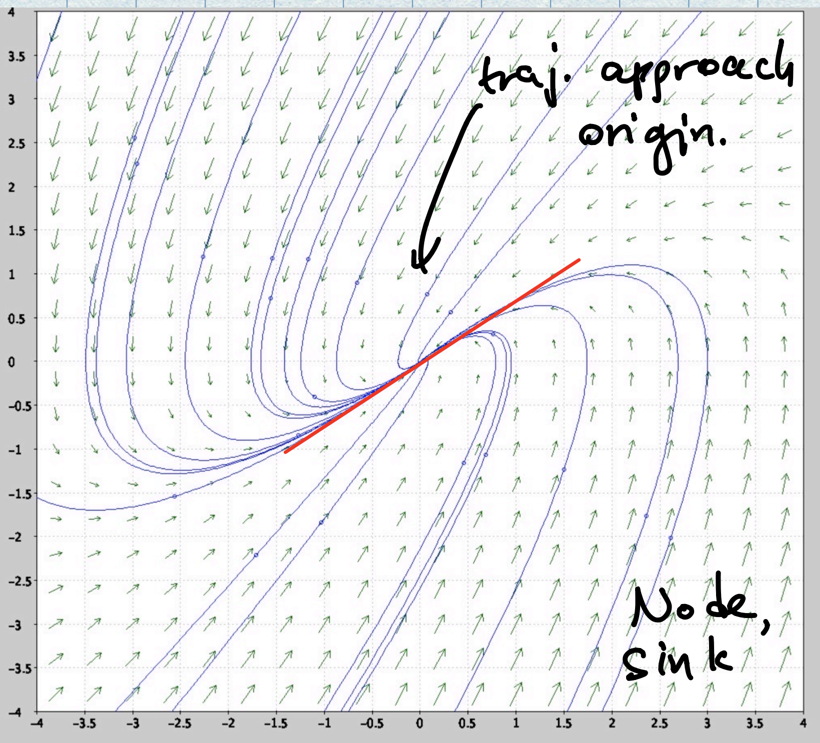
$$x' = -y$$
$$y' = x - 2y$$

Sol'n:

$$\underline{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} (-t) + \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) e^{-t}$$

as $t \to \infty$

$\underline{x}(t) \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$



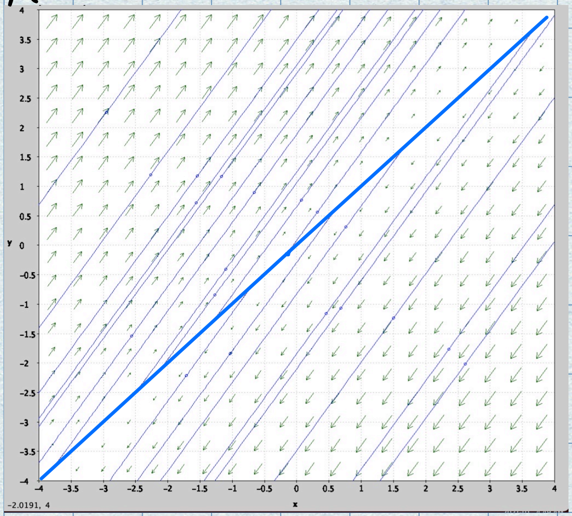
improper nodal sink.

$\downarrow$

more than 2 curves tang. to the same line @ origin

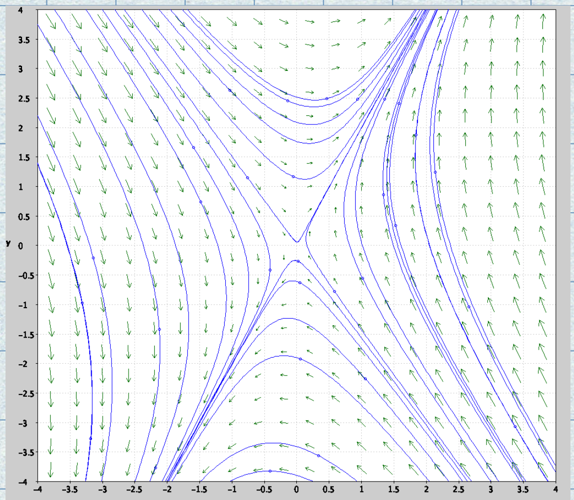
A

not a node.

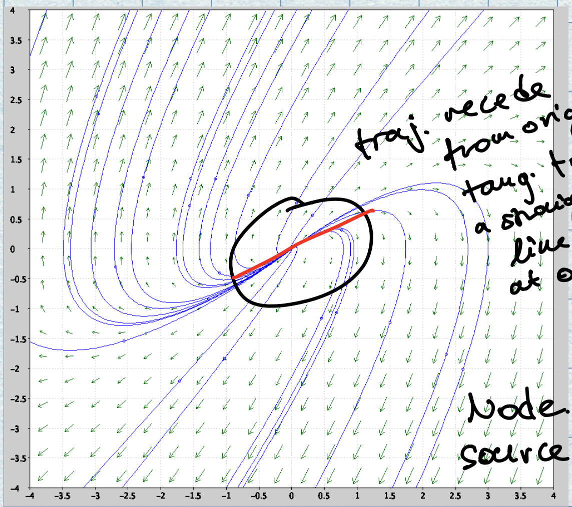


B

not node



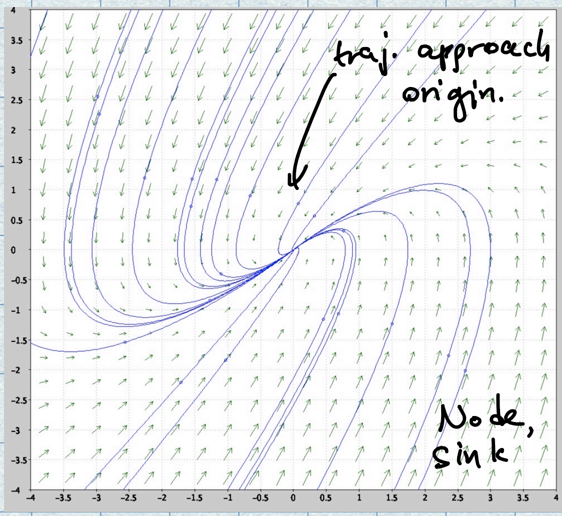
C



traj. recede from origin, tang. to a straight line at origin

Node source

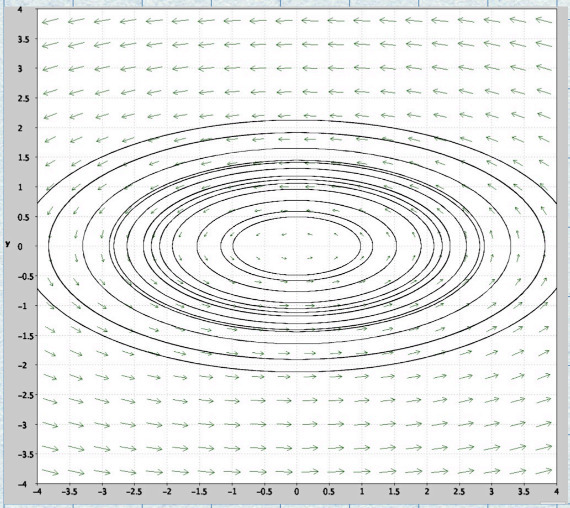
D



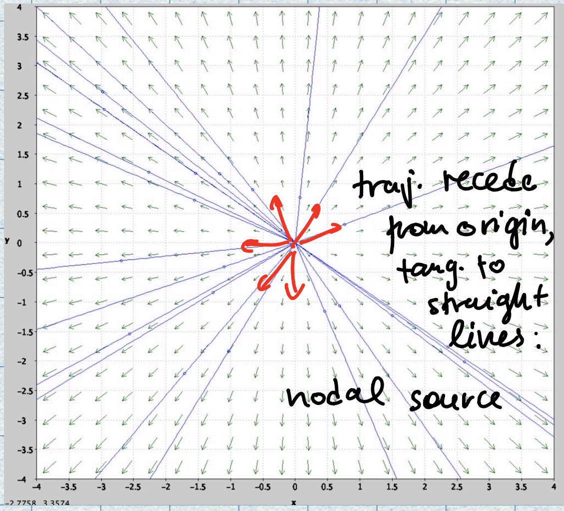
traj. approach origin.

Node, sink

E



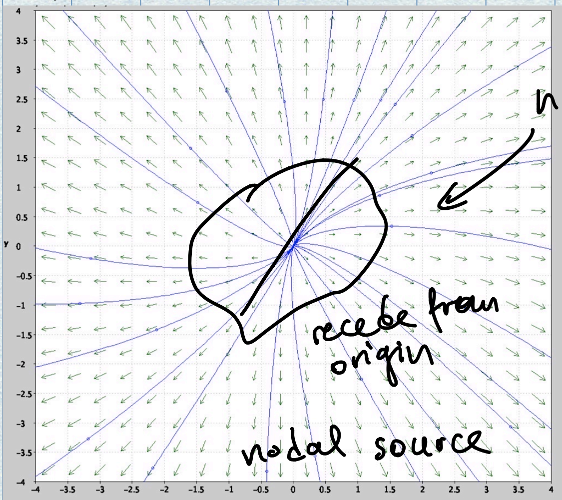
F



traj. recede from origin, tang. to straight lines.

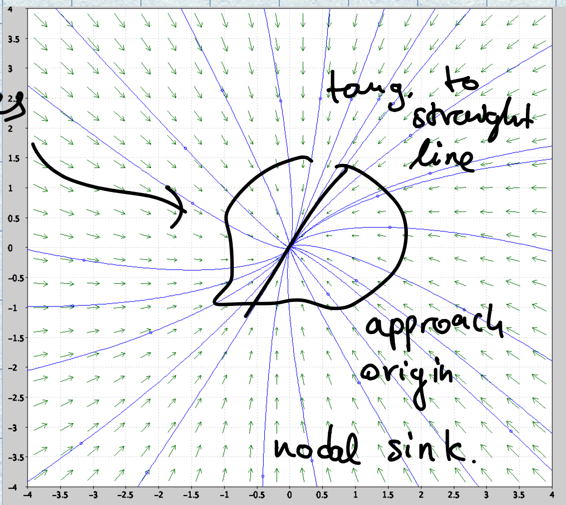
nodal source

G

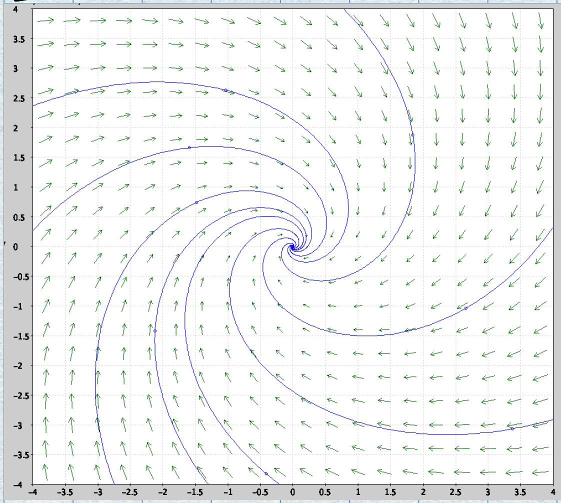

recede from
origin
nodal source

nodes

H


tang. to
straight
line
approach
origin
nodal sink.

I



J