Plan for today
5.6

Learning goals/important concepts
1. Fundamental matrix, exponential of a matrix
2. Be able to compute the exponential of a diagonal or nilpotent matrix

Reminders/Announcements
1. Read the textbook!
2. Quiz 5 grades will be ready by Monday
3. Office hours 2-3 pm today
4. HW 29 due tonight

5.6. $\underline{x}' = \overbrace{\underline{A}(t)}^{n\times n}\underline{x}$ linear systems

n lin. indep. sols $\underline{x}_1, \ldots, \underline{x}_n$

General sol'n: $\underline{x} = c_1\underline{x}_1 + \ldots + c_n\underline{x}_n$

$n\times 1$ col. vectors.

Arrange into a matrix: $\underbrace{\underline{\Phi}(t) = \overbrace{\begin{bmatrix} | & & | \\ \underline{x}_1 & \cdots & \underline{x}_n \\ | & & | \end{bmatrix}}^{\det \Phi(t) = W(\underline{x}_1, \ldots, \underline{x}_n)}}_{\text{fundamental matrix.}}$

$\underline{x} = \underline{\Phi}(t)\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Ex 1. $x_1' = 5x_1 - 4x_2$
$x_2' = 3x_1 - 2x_2$

$\underline{A} = \begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix}$

Eigen. $\lambda = 1$ e.v. $\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 2$ e.v. $\underline{v}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

2 lin. indep. sols:

$\underline{x}_1 = e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \underline{x}_2 = e^{2t}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$$\underline{\underline{\Phi}}(t) = \begin{bmatrix} e^t & 4e^{2t} \\ e^t & 3e^{2t} \end{bmatrix}$$

Note: $\underline{\underline{\Phi}}(t)\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 e^t + c_2 4e^{2t} \\ c_1 e^t + c_2 \cdot 3e^{2t} \end{bmatrix}$

$$= c_1 \begin{bmatrix} e^t \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} 4e^{2t} \\ 3e^{2t} \end{bmatrix} \quad //$$

Fund. matrix is n̲ot unique: if $\underline{\underline{\Phi}}(t)$ is a F.M. then

$$\underline{\underline{\Phi}}(t) \cdot \underbrace{\underline{\underline{C}}}_{n\times n \text{ non-singular}} \text{ is a F.M.}$$

Ex: $\begin{bmatrix} e^t & 4e^{2t} \\ e^t & 3e^{2t} \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\substack{\text{non-sing.} \\ 2\times 2 \text{ matrix}}}$

$$= \begin{bmatrix} 4e^{2t} & e^t \\ 3e^{2t} & e^t \end{bmatrix} \to \text{still a F.M.}$$

Now: IVP

$$\begin{cases} \underline{x}' = \underline{\underline{A}}(t)\,\underline{x} \\ \underline{x}(a) = \underline{x}_a \leftarrow \text{known vector.} \end{cases} \qquad (*)$$

let $\underline{\underline{\Phi}}(t)$ be a fund. matrix. ← known

$x(t) = \Phi(t)\,\underline{\underline{c}}$ → general sol'n.

$\underline{\underline{c}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ unknown.

$\underline{\underline{x}}(a) = \underline{\underline{x}}_a$ known

$$\Rightarrow \underbrace{\underline{\underline{\Phi}}(a)}_{\text{known}}\,\underline{\underline{c}} = \underbrace{\underline{\underline{x}}}_{\text{known}}{}_a$$

$$\Rightarrow \underline{\underline{c}} = \underbrace{\underline{\underline{\Phi}}(a)^{-1}}_{\text{inverse always exists.}}\underline{\underline{x}}_a$$

So: $\Phi(t)\,\Phi(a)^{-1}\,\underline{\underline{x}}_a$ is the sol'n to $\star$

Ex: $x_1' = 5x_1 - 4x_2$
$x_2' = 3x_1 - 2x_2$
$\begin{cases} x_1(0) = 5 \\ x_2(0) = 3 \end{cases}$ IVP

$\Phi(t) = \begin{bmatrix} e^t & 4e^{2t} \\ e^t & 3e^{2t} \end{bmatrix}$ $\underline{\underline{x}}_0 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

Sol'n will be:

$$\Phi(t)\,\Phi(0)^{-1}\,\underline{\underline{x}}_0 = \begin{bmatrix} e^t & 4e^{2t} \\ e^t & 3e^{2t} \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}^{-1}\begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -3e^t + 4e^{2t} & 4e^t - 4e^{2t} \\ -3e^t + 3e^{2t} & 4e^t - 3e^{2t} \end{bmatrix}}_{\star}\begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

<u>Upshot:</u> The sol'n to <u>any</u> IVP

$$x_1' = 5x_1 - 4x_2 \qquad A = \begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix}$$

$$x_2' = 3x_1 - 2x_2$$

$$\begin{cases} x_1(0) = a \\ x_2(0) = b \end{cases}$$

is

$$\underbrace{\begin{bmatrix} -3e^t + 4e^{2t} & 4e^t - 4e^{2t} \\ -3e^t + 3e^{2t} & 4e^t - 3e^{2t} \end{bmatrix}}_{\Phi(t)\Phi(0)^{-1}} \begin{bmatrix} a \\ b \end{bmatrix}$$

<u>Note:</u> If $\Phi(t)$ is a F.M. for $\underline{x}' = A\underline{x}$ (no t here.)

$\underline{\underline{X}}(t) = \underline{\underline{\Phi}}(t)\underline{\underline{\Phi}}(0)^{-1}$ is a F.M.

and it solves IVP

$$(*) \begin{cases} \underbrace{\underline{\underline{X}}'(t)}_{n\times n} = \underline{\underline{A}} \underbrace{\underline{\underline{X}}(t)}_{n\times n} \\ \underbrace{\underline{\underline{X}}(0)}_{n\times n} = \underbrace{\underline{\underline{I}}}_{n\times n} \end{cases}$$

no t

works bec. columns of $\underline{\underline{X}}$ are sols of $\underline{\underline{x}}' = \underline{\underline{A}}\underline{\underline{x}}$

<u>Ex:</u>

$$\underline{\underline{X}}(t) = \begin{bmatrix} -3e^t + 4e^{2t} & 4e^t - 4e^{2t} \\ -3e^t + 3e^{2t} & 4e^t - 3e^{2t} \end{bmatrix}$$

<u>Check:</u> $\underline{\underline{X}}'(t) = \begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix} \underline{\underline{X}}(t)$

<u>and:</u> $\underline{\underline{X}}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underline{\underline{Id}}$ //

Compare (*) w/ $\begin{cases} x' = kx \\ x(0) = 1 \end{cases}$ (k const.) $\qquad \underline{x(t) = e^{kt}}$

<u>Hope</u>: can make sense of $e^{\underline{A}t}$, and it will solve (*)

<u>Recall</u>: $e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \ldots$

Define, for a <u>matrix</u> $\underline{\underline{A}}$ $(n \times n)$

$$\underbrace{e^{\underline{\underline{A}}}}_{n \times n \text{ matrix}} = \underbrace{\underline{\underline{I}}}_{n \times n} + \underbrace{\underline{\underline{A}}}_{n \times n} + \frac{1}{2!}\underbrace{\underline{\underline{A}}^2}_{n \times n} + \frac{1}{3!}\underline{\underline{A}}^3 + \ldots$$

How do we compute $e^{\underline{\underline{A}}}$?

Start w/ Easy Cases.

→ <u>Diagonal</u> matrices.

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^2 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^k = \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \longleftarrow \text{special to diag. matrices.}$$

$$\underline{\underline{A}} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$$e^{\underline{\underline{A}}} = Id + \underline{\underline{A}} + \frac{1}{2!}\underline{\underline{A}}^2 + \frac{1}{3!}\underline{\underline{A}}^3 + \ldots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} a^3 & 0 \\ 0 & b^3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + a + \frac{1}{2!}a^2 + \ldots & 0 \\ 0 & 1 + b + \frac{1}{2!}b^2 + \ldots \end{bmatrix} = \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix}$$

Can use def'n to compute $e^{\underline{\underline{A}}}$, $\underline{\underline{A}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

$$e^{\underline{\underline{A}}t} = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \qquad \left(\underline{\underline{A}}t = \begin{bmatrix} t & 0 \\ 0 & 2t \end{bmatrix} \right)$$