Plan for today:

5.6

Learning goals:

1. Be able to compute matrix exponentials for diagonal matrices and nilpotent matrices
2. Be able to compute a matrix exponential of a matrix that can be written as a sum of a multiple of the identity and a nilpotent matrix
3. Be able to compute exponential matrices by solving the corresponding system of ode

Announcements

1. No class on Wednesday (will be held as office hours, in OH meeting link)
2. Quiz grades will be posted later today

Last time: Fundamental Matrices.

$\underline{x}' = \underset{=}{\overset{n \times n}{A}} \underline{x}$

arrange n lin. indep. sols into an $n \times n$ matrix.

If $\underset{=}{\Phi}(t)$ is a F.M. for $\underline{x}' = \underset{=}{A}\underline{x}$

$\underset{=}{X}(t) = \underset{=}{\Phi}(t)\underset{=}{\Phi}(0)^{-1}$ is a F.M.

and it solves IVP

$$(*) \begin{cases} \underset{=}{X}'(t) = \underset{=}{A} \underset{=}{X}(t) \\ \underset{=}{X}(0) = \underset{=}{I} \end{cases}$$

no t

hoped to find $\underset{=}{X}(t)$ as $e^{\underset{=}{A}t}$

$\begin{cases} y' = ky \\ y(0) = 1 \end{cases}$ e^{kt}

Ex: $x_1' = 5x_1 - 4x_2$
$x_2' = 3x_1 - 2x_2$

$\underset{=}{A} = \begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix}$

$$\Phi(t) = \begin{bmatrix} e^t & 4e^{2t} \\ e^t & 3e^{2t} \end{bmatrix},$$

$$\Phi(t)\Phi(0)^{-1} = \begin{bmatrix} -3e^t + 4e^{2t} & 4e^t - 4e^2 \\ -3e^t + 3e^{2t} & 4e^t - 3e^2 \end{bmatrix} //$$

Computing Matrix exponentials

$$e^{\underline{\underline{A}}} = \underline{\underline{I}} + \underline{\underline{A}} + \frac{1}{2}\underline{\underline{A}}^2 + \frac{1}{3!}\underline{\underline{A}}^3 + \dots \qquad (*)$$

Saw: diagonal mat. $\underline{\underline{A}} = \begin{pmatrix} a_1 & 0 & & 0 \\ 0 & a_2 & & 0 \\ & & \ddots & \\ 0 & 0 & & a_n \end{pmatrix}$

$$e^{\underline{\underline{A}}} = \begin{pmatrix} e^{a_1} & & \\ & e^{a_2} & 0 \\ 0 & & \ddots \\ & & & e^{a_n} \end{pmatrix}$$

Nilpotent matrices: means $\underline{\underline{A}}^k = \underline{\underline{0}}$ for some integer k.

Note: $\underline{\underline{A}}^p = \underline{\underline{0}}$ for any $p \geq k$

Ex: $\underline{\underline{A}}_2 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$, $\underline{\underline{A}}_2^2 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$(*) \Rightarrow e^{\underline{\underline{A}}_2} = I + \underline{\underline{A}}_2 + \frac{1}{2}\cancel{\underline{\underline{A}}_2^2} + \cancel{\dots}$ 0 0

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

! $e^{\underline{\underline{A}}}$ is <u>not</u> given by raising raising e to the entries of $\underline{\underline{A}}$

$$e^{\underline{\underline{A}}_2} \neq \begin{bmatrix} 0 & e^2 \\ 0 & 0 \end{bmatrix}$$

Properties

HW

① If $\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{B}}\,\underline{\underline{A}}$ then $e^{\underline{\underline{A}}+\underline{\underline{B}}} = e^{\underline{\underline{A}}} e^{\underline{\underline{B}}}$

Ex: $\underline{\underline{B}} = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}$

hope: $\underline{\underline{B}} = \underline{\underline{B}}_1 + \underline{\underline{B}}_2$

so that $\underline{\underline{B}}_1 \underline{\underline{B}}_2 = \underline{\underline{B}}_2 \underline{\underline{B}}_1$

and can compute $e^{\underline{\underline{B}}_1}$, $e^{\underline{\underline{B}}_2}$

diagonal ← $\underline{\underline{B}}_1 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = 3\,\underline{\underline{I}}$

nilpotent ← $\underline{\underline{B}}_2 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$

Check: $\underline{\underline{B}}_1 \underline{\underline{B}}_2 = 3\,\underline{\underline{I}} \cdot \underline{\underline{B}}_2 = 3\,\underline{\underline{B}}_2\,\underline{\underline{I}} = \underline{\underline{B}}_2 (3\,\underline{\underline{I}})$
$= \underline{\underline{B}}_2 \underline{\underline{B}}_1$

By ① $e^{\underline{\underline{B}}} = e^{\underline{\underline{B}}_1 + \underline{\underline{B}}_2} = e^{\underline{\underline{B}}_1} e^{\underline{\underline{B}}_2} = \begin{bmatrix} e^3 & 0 \\ 0 & e^3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} e^3 & 2e^3 \\ 0 & e^3 \end{bmatrix}$$

previous ex.

In ① $\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{B}}\,\underline{\underline{A}}$:

take $\underline{\underline{A}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $\underline{\underline{B}} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$

check: $e^{\underline{\underline{A}}} e^{\underline{\underline{B}}} \neq e^{\underline{\underline{B}}} e^{\underline{\underline{A}}}$

not both can be the same as $e^{\underline{\underline{A}}+\underline{\underline{B}}}$.

(2) $e^{\underline{\underline{0}}} = \underline{\underline{I}}$ (similar to $e^0 = 1$)

(3) $\left(e^{\underline{\underline{A}}}\right)^{-1} = e^{-\underline{\underline{A}}}$ ($e^{\underline{\underline{A}}}$ always invertible)

matrix inverse

expon. of $-\underline{\underline{A}}$

(4) $\frac{d}{dt} e^{\underline{\underline{A}}t} = \underline{\underline{A}} e^{\underline{\underline{A}}t}$ (compare w/ $(e^{kt})' = ke^{kt}$

($\underline{\underline{A}}$ $n \times n$ const. coef. matrix)

So: by (2), (4):

$$\underline{\underline{X}}(t) = e^{\underline{\underline{A}}t}$$

$\underline{\underline{A}}$ const. coef.

solves

$$\begin{cases} \underline{\underline{X}}'(t) = \underline{\underline{A}} \, X(t) & [9] \\ \underline{\underline{X}}(0) = \underline{\underline{I}} \end{cases}$$

IVP we saw earlier.

So:

$$\boxed{e^{\underline{\underline{A}}t} = \Phi(t)\,\Phi(0)^{-1},}$$ (*)

Φ fund. matrix for $x' = Ax$

Use (*) in 2 ways:

(i): If $e^{\underset{=}{A}t}$ is known, gives a F.M.

Ex: Compute: if $\underset{=}{A} = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}$

check: $e^{\underset{=}{A}t} = \begin{bmatrix} e^{3t} & 2te^{3t} \\ 0 & e^{3t} \end{bmatrix}$ exercise; break into diagonal & nilpotent.

So; a F.M for $\left(\begin{array}{l} x_1' = 3x_1 + 2x_2 \\ x_2' = 3x_2 \end{array} \right)$ $x' = \underset{=}{A} x$ is $\tilde{\Phi}(t) = \begin{bmatrix} e^{3t} & 2te^{3t} \\ 0 & e^{3t} \end{bmatrix}$

If $\begin{cases} x' = \underset{=}{A} x \\ \underset{=}{x(0)} = \begin{bmatrix} a \\ b \end{bmatrix} \end{cases}$ then sol'n

$$\underset{=}{x} = e^{\underset{=}{A}t} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a e^{3t} + b \cdot 2te^{3t} \\ b e^{3t} \end{bmatrix}$$

(ii) If a FM is known we can compute $e^{\underset{=}{A}t}$.

Ex: $x_1' = 5x_1 - 4x_2$ $\quad \underset{=}{A} = \begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix}$
$x_2' = 3x_1 - 2x_2$

$$\Phi(t) = \begin{bmatrix} e^t & 4e^{2t} \\ e^t & 3e^{2t} \end{bmatrix},$$

$$\Phi(t)\Phi(0)^{-1} = \begin{bmatrix} -3e^t + 4e^{2t} & 4e^t - 4e^2 \\ -3e^t + 3e^{2t} & 4e^t - 3e^2 \end{bmatrix}.$$

So:
$$e^{\underline{A}t} = \begin{bmatrix} -3e^{t}+4e^{2t} & 4e^{t}-4e^{2} \\ -3e^{t}+3e^{2t} & 4e^{t}-3e^{2} \end{bmatrix}.$$

Method of computing $e^{\underline{A}t}$ for complicated matrices:

1. solve $\underline{x}' = \underline{A}\,\underline{x}$ ($\underline{A}$ $n \times n$, const. coef.)
2. find n lin. indep. sols
3. arrange into matrix to find F. M. $\varphi(t)$.
4. $e^{\underline{A}t} = \varphi(t)\,\varphi^{-1}(0)$

5.7. Non-homog. eqs

$$\underline{x}' = \underline{A}(t)\underline{x} + \underline{f}(t)$$

Gen. sol'n: $\underline{x} = \underline{x}_c(t) + \underline{x}_p$

sol'n of $\underline{x}' = \underline{A}(t)\,\underline{x}$

part. sol'n

(1) Undetermined Coefficients

Want: $\underline{A}$ const. coef. matrix.

$\underset{=}{f} \to$ lin. comb. of products of 1. polyn.
2. exp.
3. $\cos(kx)$, $\sin(kx)$

<u>Ex</u>

$$\underset{=}{f}(t) = \begin{bmatrix} t\sin(t) \\ e^t + 2e^{2t} \end{bmatrix} = t\sin(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + e^{2t}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

const. coef.

<u>Non-ex.</u>

$$\begin{bmatrix} \tan(t) \\ e^t \end{bmatrix}$$ does not work

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