Plan for today:
5.7
Start 7.1

Learning Goals:
1. Be able to solve nonhomogeneous linear systems using undetermined coefficients/ variation of parameters
2. Know the definition of the Laplace transform

Announcements:
1. Final exam will be available online from 7:00pm of May 4 to 7:00 pm of May 5\
2. Quiz 6 (last) next Thursday. Covers 5.5, 5.3, 5.6, 5.7
3. Read the textbook

5.7 Non-homog. systems.

$$\underline{\underline{x}}' = \underline{\underline{A}}(t)\underline{\underline{x}} + \underline{\underline{f}}$$

$$\underline{\underline{x}} = \underline{\underline{x}}_c + \underline{\underline{x}}_p$$

part. sol'n.

$$\underline{\underline{x}}_c' = \underline{\underline{A}}(t)\underline{\underline{x}}_c$$

Undetermined Coef. (for systems)

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \underbrace{\begin{bmatrix} 9 & 1 \\ -8 & -2 \end{bmatrix}}_{\text{const. coef.}} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e^t \\ te^t \end{bmatrix} \qquad (*)$$

Compl. sol'n:

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 9 & 1 \\ -8 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det \begin{bmatrix} 9-\lambda & 1 \\ -8 & -2-\lambda \end{bmatrix} = 0 \Rightarrow \lambda_\pm = \frac{7 \pm \sqrt{89}}{2}$$

find.

Find eigenvectors for $\lambda_\pm$,

$$\underline{x}_c = e^{\frac{7+\sqrt{89}}{2}t}\,\underline{v}_1 + e^{\frac{7-\sqrt{89}}{2}t}\,\underline{v}_2$$

eigenvectors.

Part. sol'n: Find nice building blocks

$$\underline{f}(t) = e^t\begin{bmatrix}1\\0\end{bmatrix} + te^t\begin{bmatrix}0\\1\end{bmatrix}$$

Building blocks: linear comb. of functions appearing in $\underline{f}$ & their derivatives.

$$x_p = e^t\begin{bmatrix}c_1\\c_2\end{bmatrix} + te^t\begin{bmatrix}b_1\\b_2\end{bmatrix}$$

undet. coef.

Duplication? No!

So x_p is a good guess.

Plug in x_p into (*), find c_1, c_2, b_1, b_2

Sol'n at the end

Difference when there is duplication

from ch3.

Ex: $y'' - 2y' + y = e^t$

$y_c = c_1 e^t + c_2 te^t$

guess: $y = ce^t$ overlap

$y_p = c\,t^2 e^t$ determine c.

multiply by power of t.

seen before

Now

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t e^{2t} \\ 0 \end{bmatrix} \quad (*)$$

Comp. sol'n: $\underline{\underline{x}}_c = e^{2t} \underline{\underline{v}}_1 + e^{-3t} \underline{\underline{v}}_2$ (check)

eigenvectors

Guess for $\underline{\underline{x}}_p$

$$\underline{f}(t) = \begin{bmatrix} te^{2t} \\ 0 \end{bmatrix} = te^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\underline{\underline{x}}_1 = e^{2t} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + te^{2t} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

unknown constants

Duplication?

Multiply $\underline{\underline{x}}$ by lowest power of t for which there is no duplication in any term.

t is good!

$$t\underline{\underline{x}}_1 = te^{2t} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + t^2 e^{2t} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

no duplication

Difference: In the actual guess

include lower powers of t:

$$\underset{=}{x}_p = \underbrace{e^{2t}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\text{surprise, different from before.}} + te^{2t}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + t^2e^{2t}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Review problem 1 is relevant.
plug in to $*$, find $a_1, a_2, b_1, b_2, c_1, c_2$

For exams: Undetermined coef. is good only when right hand side is simple.

Variation of Parameters

$$\underset{=}{x}' = \underset{=}{P}(t)\underset{=}{x} + \underset{=}{f}(t) \qquad *$$

cont.

can depend on t
(not the case for undet. coef.)

If $\Phi(t)$ is a fund. matrix for $\underset{=}{x}' = \underset{=}{P}(t)\underset{=}{x}$
then a particular sol'n to $*$ is

$$\underset{=}{x}_p = \underset{=}{\Phi}(t)\int \underset{=}{\Phi}(t)^{-1}\underset{=}{f}(t)\,dt$$

indefinite integral.

Check:

$$\underline{\underline{x}}_p' = \underline{\underline{\Phi}}'(t) \int \underline{\underline{\Phi}}(t)^{-1} \underline{f}(t)\, dt + \underline{\underline{\Phi}}(t) \underbrace{\cancel{\underline{\underline{\Phi}}(t)^{-1}}}_{I} \underline{f}(t)$$

$$\underline{\underline{\Phi}}' = \underline{\underline{P}}(t)\underline{\underline{\Phi}}(t)$$

$$= \underline{\underline{P}}(t)\, \underline{\underline{\Phi}}(t) \int \underline{\underline{\Phi}}(t)^{-1} \underline{f}(t)\, dt + \underline{f}(t)$$

$$= \underline{\underline{P}}(t)\, \underline{x}_p + \underline{f}(t)$$

If we know $\Phi(t)$ we are happy.

→ If $\underline{x}' = \underline{\underline{A}}\,\underline{x} + \underline{f}$, $\underline{\underline{A}}$ const. coef, $\Phi(t) = e^{\underline{\underline{A}}t}$ is a F.M.

Sol'n to IVP

$$\begin{cases} \underline{x}' = \underline{\underline{A}}\,\underline{x} + \underline{f} \\ \underline{x}(0) = \underline{x}_0 \end{cases}$$

definite integral

is

$$\underline{x} = \underbrace{e^{\underline{\underline{A}}t}\, \underline{x}_0}_{\text{compl. sol'n.}} + e^{\underline{\underline{A}}t} \int_0^t e^{-\underline{\underline{A}}s}\, \underline{f}(s)\, ds$$

$$\left[\underline{x} = \underline{e^{\underline{\underline{A}}t}\, x_0} + \int_0^t e^{A(t-s)} f(s)\, ds \right]$$

Compare: $\underline{x}_p = \underline{\underline{\Phi}}(t) \int \underline{\underline{\Phi}}(t)^{-1} \underline{f}(t)\, dt$

$$\underline{\text{Ex:}} \quad \begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} te^{2t} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\underline{\text{Given:}} \quad e^{At} = \frac{1}{5} \begin{bmatrix} e^{-3t} + 4e^{2t} & -2e^{-3t} + 2e^{2t} \\ -2e^{-3t} + 2e^{2t} & 4e^{-3t} + e^{2t} \end{bmatrix}$$

$$\underline{x} = \frac{1}{5} \begin{bmatrix} e^{-3t} + 4e^{2t} & -2e^{-3t} + 2e^{2t} \\ -2e^{-3t} + 2e^{2t} & 4e^{-3t} + e^{2t} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$+ \int_0^t \frac{1}{5} \begin{bmatrix} e^{-3(t-s)} + 4e^{2(t-s)} & -2e^{-3(t-s)} + 2e^{2(t-s)} \\ -2e^{-3(t-s)} + 2e^{2(t-s)} & 4e^{-3(t-s)} + e^{2(t-s)} \end{bmatrix} \begin{bmatrix} se^{2s} \\ 0 \end{bmatrix} ds$$

$$= \frac{1}{5} \begin{bmatrix} -4e^{-3t} + 4e^{2t} \\ 8e^{-3t} + 2e^{2t} \end{bmatrix}$$

$$+ \int_0^t \frac{1}{5} \begin{bmatrix} se^{2s} \left(e^{-3(t-s)} + 4e^{2(t-s)} \right) \\ se^{2s} \left(-2e^{-3(t-s)} + 2e^{2(t-s)} \right) \end{bmatrix} ds$$

//

Sol'n: Ex 1. Undetermined Coefficients

Ex:

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \underbrace{\begin{bmatrix} 9 & 1 \\ -8 & -2 \end{bmatrix}}_{\underline{\underline{A}}} \begin{bmatrix} x \\ y \end{bmatrix} + \underbrace{\begin{bmatrix} e^t \\ te^t \end{bmatrix}}_{\underline{\underline{f}}(t)} \quad (1)$$

Find Eigenvalues $\lambda = \dfrac{7 \pm \sqrt{89}}{2}$

Comp'l. Sol'n:

$$\underline{\underline{x}}_c = \underline{\underline{v}}_1 e^{\frac{7+\sqrt{89}}{2}t} + \underline{\underline{v}}_2 e^{\frac{7-\sqrt{89}}{2}t}$$

where $\underline{\underline{v}}_1, \underline{\underline{v}}_2$ are eigenvectors associated w/ $\lambda_1 = \dfrac{7+\sqrt{89}}{2}$, $\lambda_2 = \dfrac{7-\sqrt{89}}{2}$

$$\underline{\underline{f}}(t) = e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + te^t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\underline{\underline{x}}_p = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} te^t$$

No overlap of building blocks

Substitute $\underline{\underline{x}}_p$ into (1):

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} e^t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} te^t$$

$$= A \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^t + A \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} te^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} te^t$$

Note: $A \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 9a_1 + a_2 \\ -8a_1 - 2a_2 \end{bmatrix}$, $A \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 9b_1 + b_2 \\ -8b_1 - 2b_2 \end{bmatrix}$

Equate Coefficients

Coef. of e^t: $a_1 + b_1 = 9a_1 + a_2 + 1$

e^t: $a_2 + b_2 = -8a_1 - 2a_2$

te^t: $b_1 = 9b_1 + b_2$

te^t: $b_2 = -8b_1 - 2b_2 + 1$

Collect terms:

$$\begin{array}{cccccc} -8a_1 & -a_2 & +b_1 & & = & 1 \\ 8a_1 & +3a_2 & + & b_2 & = & 0 \\ 0 & 0 & -8b_1 & -b_2 & = & 0 \\ 0 & 0 & 8b_1 & 3b_2 & = & 1 \end{array}$$

Let $\underline{\underline{B}} = \begin{bmatrix} -8 & -1 & 1 & 0 \\ 8 & 3 & 0 & 1 \\ 0 & 0 & -8 & -1 \\ 0 & 0 & 8 & 3 \end{bmatrix}$

Can use a CAS to find $\begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix} = \underline{\underline{B}}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Or: Row reduction:

$$\left[\begin{array}{cccc|c} -8 & -1 & 1 & 0 & 1 \\ 8 & 3 & 0 & 1 & 0 \\ 0 & 0 & -8 & -1 & 0 \\ 0 & 0 & 8 & 3 & 1 \end{array}\right]$$

$$\begin{array}{c} (1)+(2)\to(2) \\ \longrightarrow \\ (3)+(4)\to(4) \end{array} \left[\begin{array}{cccc|c} -8 & -1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & -8 & -1 & 0 \\ 0 & 0 & 0 & 2 & 1 \end{array}\right]$$

$$\Rightarrow \quad b_2 = \frac{1}{2}, \quad -8b_1 - b_2 = 0 \Rightarrow b_1 = -\frac{1}{16}$$

$$2a_2 + b_1 + b_2 = 1 \Rightarrow \quad a_2 = \frac{9}{32}$$

$$-8a_1 - a_2 + b_1 = 1 \Rightarrow a_1 = -\frac{1}{8}\left(1 + \frac{9}{32} + \frac{1}{16}\right)$$

$$\rightarrow a_1 = -\frac{43}{256}$$

So:

$$x_p(t) = \begin{bmatrix} -\frac{43}{256} \\ \frac{9}{32} \end{bmatrix} e^t + \begin{bmatrix} -\frac{1}{16} \\ \frac{1}{2} \end{bmatrix} te^t$$