Plan for today:

7.1- start 7.2

Learning Goals

1. Be able to compute the Laplace Transform from the definition
2. Be able to compute the inverse Laplace transform by breaking up a given function F(s) into a linear combination of functions for which the inverse Laplace transform is known.
3. Know the rule on differentiation and be able to apply it to solve IVPs

Announcements-Reminders

1. Read the textbook!
2. Table of Laplace Transforms available on the MA 266 course website

Laplace Transform

Tool for solving linear const. coef. eqs:

Ex: $mx'' + cx' + kx = g(t)$ (spring-mass system)

const.

Laplace tr. works well when $g(t)$ is not cont. on $[0,\infty)$.

Defn: $f(t)$ given defined for $t \geq 0$. Its Laplace tr. is given by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \lim_{M\to\infty} \int_0^M e^{-st} f(t)\,dt$$

for s for which converges.

functions of $t \xrightarrow{\mathcal{L}}$ functions of s

Ex 1: Exponentials. $f(t) = e^{at}$, $t \geq 0$

a real for now

$$\mathcal{L}\{e^{at}\} = \lim_{M\to\infty} \int_0^M e^{-st} e^{at}\, dt \quad (*)$$

$$= \lim_{M\to\infty} \int_0^M e^{(a-s)t}\, dt$$

$$= \lim_{M\to\infty} \left. \frac{e^{(a-s)t}}{a-s} \right|_0^M$$

$$= \lim_{M\to\infty} \left(\frac{e^{(a-s)M}}{a-s} - \frac{1}{a-s} \right)$$

Note: $a > s \Rightarrow (a-s) > 0$

$$\lim_{M\to\infty} \frac{e^{(a-s)M}}{a-s} = \infty \quad \text{bad!}$$

If $a = s$ $\quad \frac{e^{(a-s)M}}{a-s}$ not defined

$a < s \Rightarrow (a-s) < 0$

$$\lim_{M\to\infty} \frac{e^{(a-s)M}}{a-s} = 0$$

So: For $s > a$ $(*)$ converges and

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \quad (*)$$

Note: $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ if a cplx (check!) and makes sense for $s > \mathrm{Re}(a)$.

Rmk: Makes sense for exponential functions;

and if $|f(t)| \le Me^{ct}$ for some M, c
then

↳ of exponential order.

$\mathcal{L}\{f(t)\}$ is defined for $s > c$.

$\mathcal{L}$ can handle functions growing fast.

<u>Ex 2:</u> set $a = 0$ in (*)

$$\mathcal{L}\{1\} = \frac{1}{s}$$

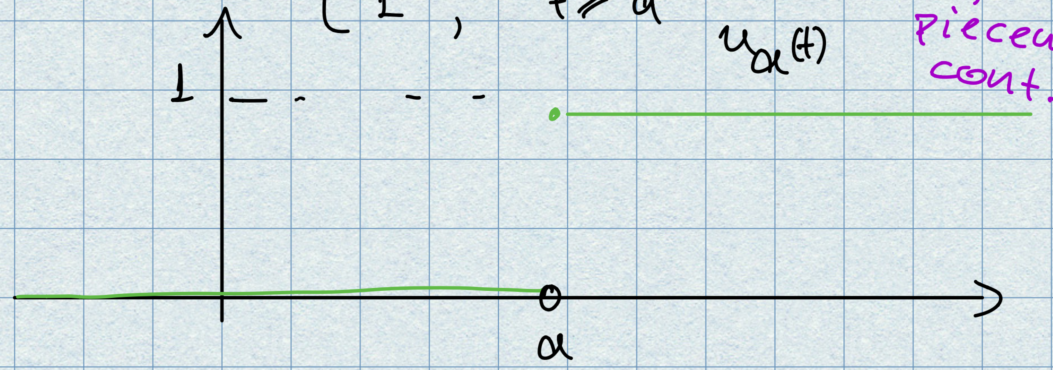
<u>Ex 3:</u> Step functions:

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

unit step fct. (or Heaviside function)

$$u_a(t) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

← example of a piecewise cont. function



$a > 0$:

$$\mathcal{L}\{u_a(t)\} = \lim_{M\to\infty} \int_0^M e^{-st} u_a(t)\, dt$$

$$= \lim_{M\to\infty} \int_a^M e^{-st}\, dt$$

← bec. $u_a(t) = 0$ for $t < a$

$$= \frac{e^{-as}}{s} \quad \text{for } s > 0.$$

Read: p. 439 on Gamma function.

Laplace tr. is linear!

$$\mathcal{L}\{a f(t) + b g(t)\} = a \mathcal{L}\{f(t)\} + b \mathcal{L}\{g(t)\}$$

constants.

Ex: $\mathcal{L}\{3t^4 + 5\cosh(3t)\}$

$$= 3\mathcal{L}\{t^4\} + 5\mathcal{L}\{\cosh(3t)\}$$

$$\stackrel{\text{Laplace table}}{=} 3 \cdot \frac{4!}{s^5} + 5 \frac{s}{s^2 - 9}.$$

The table: https://www.math.purdue.edu/academic/files/courses/2013spring/MA26600/LT.pdf

If $F(s) = \mathcal{L}\{f(t)\}$ then $f(t) = \mathcal{L}^{-1}\{F(s)\}$ is the inverse Laplace transform.

Process: Write $F(s)$ as a sum of functions for which $\mathcal{L}^{-1}$ is given by table.

Ex:

$$F(s) = \frac{1}{s(s^2+4s+3)}, \quad \text{find } f(t) = \mathcal{L}^{-1}\{F(s)\}$$

$$= \frac{1}{s(s+3)(s+1)}$$

$$\frac{1}{s(s+3)(s+1)} \overset{\text{partial}}{\underset{\text{fractions}}{=}} \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+1}$$

use table.

<u>Find A:</u> Multiply by s, set $s=0$

$$\frac{1}{(s+3)(s+1)} = A + \frac{sB}{s+3} + \frac{s\cdot C}{s+1}$$

$$\overset{s=0}{\Longrightarrow} \quad A = \frac{1}{3}$$

<u>Find B:</u> Multiply by $(s+3)$, set $s=-3$

$$\Rightarrow B = \frac{1}{6}$$

similarly $C = -\frac{1}{2}$

$$F(s) = \frac{1}{3}\cdot\frac{1}{s} + \frac{1}{6}\frac{1}{s+3} - \frac{1}{2}\frac{1}{s+1} \Rightarrow$$

$$\boxed{\mathcal{L}^{-1}\{F(s)\} \overset{\text{Laplace table}}{=} \frac{1}{3}\cdot 1 + \frac{1}{6}e^{-3t} - \frac{1}{2}e^{-t}}$$

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<u>Crucial Property of $\mathcal{L}$</u>

$$\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$$

Turns differentiation into multiplication.

Ex:

$$\begin{cases} 4x' + 3x = 1 & \textcircled{*} \\ x(0) = 0 \end{cases}$$

Apply $\mathcal{L}$ to $\textcircled{*}$ on both sides

algebraic eq'n for $X(s)$

$$4(sX(s) - \underset{x(0)}{\underset{\shortparallel}{0}}) + 3\overset{\mathcal{L}\{x(t)\}}{\overset{\shortparallel}{\overbrace{X(s)}}} = \frac{1}{s}$$

$$\Rightarrow X(s)(4s+3) = \frac{1}{s}$$

$$\Rightarrow X(s) = \frac{1}{s(4s+3)}$$

$$\Rightarrow x(t) = \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s(4s+3)}\right\}}_{\text{use p. fractions.}}$$

Exercise.

Sol'n:

write $\dfrac{1}{s(4s+3)} = \dfrac{A}{s} + \dfrac{B}{4s+3}$

Multiply by s, set $s = 0 \Rightarrow A = \frac{1}{3}$

Multiply by $4s+3$, set $s = -\frac{3}{4} \Rightarrow B = -\frac{4}{3}$

So

$$x(t) = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{3}{4}}\right\}$$

$$= \frac{1}{3} - \frac{1}{3}e^{-\frac{3}{4}t}$$