Plan for today
7.2-7.3

Learning goals:

1. Be able to solve IVPs for linear constant coefficient equations by taking the Laplace transform on both sides and using the differentiation rule
2. Be able to expand in partial fractions
3. Know the rule for the Laplace transform of integrals
4. Know the rule for the inverse Laplace transform of a translated function of s

Reminders

1. Last quiz tomorrow
2. Read the textbook

Laplace tr.: $f(t)$, $t \geq 0$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

Formula for differentiation

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt \overset{IBP}{=}$$

$$= \underbrace{\lim_{t\to\infty} e^{-st} f(t)}_{} - f(0) + s\overbrace{\int_0^\infty e^{-st} f(t)\,dt}^{\mathcal{L}\{f(t)\}}$$

assume that f is nice enough & s is large enough that limit is 0.

$$\boxed{\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)} \quad (*)$$

Solve IVPs using (*)

Ex 1: $\begin{cases} x'' + 9x = 1 \\ x(0) = 0, \quad x'(0) = 1 \end{cases}$ (*)

[can do w/ characteristic eqn & undeterm. coef.]

$$X(s) = \mathcal{L}\{x(t)\}$$

$$\mathcal{L}\{x''(t)\} = s\,\mathcal{L}\{x'(t)\} - x'(0)$$
$$= s\left(s\,\mathcal{L}\{x(t)\} - x(0)\right) - x'(0)$$
$$= s^2\,\mathcal{L}\{x(t)\} - s\,x(0) - x'(0)$$

Apply $\mathcal{L}$ to $(*)$

$$\underbrace{s^2 X(s) - s\,\cancel{x(0)}^{\,0} - \overbrace{x'(0)}^{\overset{1}{=}}}_{\mathcal{L}\{x''\}} + \underbrace{9X(s)}_{\mathcal{L}\{9x\}} = \frac{1}{s}$$

$$(s^2+9)\,X(s) = \frac{1}{s} + 1$$

$$\Rightarrow \quad X(s) = \frac{1}{s(s^2+9)} + \frac{1}{(s^2+9)}$$

$$\Rightarrow \quad x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+9)}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} \qquad (1)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{1}{3}\,\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = \frac{1}{3}\sin(3t) \qquad (2)$$

$$\frac{1}{\underbrace{s}_{\text{linear factor}}\underbrace{(s^2+9)}_{\text{irreducible quadratic factor}}} \overset{\text{Partial Fractions}}{=} \frac{A}{s} + \frac{Bs+C}{s^2+9}$$

← bec. s^2+9 is irreducible quadr.

(can't factor as $(s-a)(s-b)$ w/ a, b reals)

$$= \frac{As^2 + 9A + Bs^2 + Cs}{s(s^2+9)}$$

$$\Rightarrow \begin{cases} A + B = 0 \\ C = 0 \\ 9A = 1 \end{cases} \quad \Rightarrow \quad \begin{aligned} A &= \tfrac{1}{9} \\ B &= -\tfrac{1}{9} \\ C &= 0 \end{aligned}$$

$$\frac{1}{s(s^2+9)} = \frac{1}{9}\frac{1}{s} - \frac{1}{9}\frac{s}{s^2+9}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+9)}\right\} = \frac{1}{9} - \frac{1}{9}\cos(3t) \quad 3$$

$$x(t) = \frac{1}{3}\sin(3t) - \frac{1}{9}\cos(3t) + \frac{1}{9} \quad //$$

Partial Fractions (7.3)

Rational fct:

$$R(s) = \frac{P(s)}{Q(s)} \quad \deg P(s) < \deg(Q(s))$$

Ex: $R(s) = \dfrac{s^2-3s+1}{s^4-2s^3+4}$

if $R = \dfrac{s^2-2s+1}{s+4}$ we can use long division to write

$$R(s) = \underbrace{\tilde{P}(s)}_{\text{polynomial}} + \underbrace{\tilde{R}(s)}_{\text{rational w/ deg(numerator) < deg(denom).}}$$

1. Factor $Q(s)$ into linear factors $(s-a)^n$ & irreducible quadratic factors $((s-a)^2+b^2)^m$

Ex: if $Q(s) = s^3 + 9s = \underbrace{s}_{\text{linear}}\underbrace{(s^2+9)}_{\text{irredecible quadr.}}$

→ Part of dec. cor. to $(s-a)^n$

$$\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \cdots + \frac{A_n}{(s-a)^n}$$

$\to$ Part of dec. $\left((s-a)^2+b^2\right)^m$

$$\frac{A_1 s + B_1}{\left((s-a)^2+b^2\right)} + \cdots + \frac{A_m s + B_m}{\left((s-a)^2+b^2\right)^n}$$

Ex: $$\frac{s-1}{(s+1)(s^2-s-2)} = \frac{s-1}{(s+1)(s+1)(s-2)} = \frac{s-1}{(s+1)^2(s-2)}$$

$$\frac{s-1}{(s+1)^2(s-2)} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s-2)}$$

For C: multiply by $(s-2)$, set $s=2$, $\Rightarrow C = \frac{1}{9}$

For B: Multiply by $(s+1)^2$, set $s=-1$

$$\frac{s-1}{s-2} = A(s+1) + B + C\frac{(s+1)^2}{(s-2)} \quad (*)$$

$\overset{s=-1}{\Rightarrow}$ $B = \frac{-2}{-3} = \frac{2}{3}$

For A:

One way: plug in $B = \frac{2}{3}$, $C = \frac{1}{9}$.

Another: differentiate $(*)$, plug in $s=-1$

$$-\frac{1}{(s-2)^2} = A + \underbrace{C\Big(2(s+1)\Big)\frac{1}{s-2} + C(s+1)^2\Big(\frac{1}{s-2}\Big)'}_{\text{go away when } s=-1}$$

$s=-1$

$$\Rightarrow \quad A = -\frac{1}{9} \quad //$$

Laplace of integrals

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\} = \frac{F(s)}{s}$$

or: $$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t \mathcal{L}^{-1}\{F(s)\}(\tau)\,d\tau = \int_0^t f(\tau)\,d\tau.$$

↳ helps remove s from denom.

Ex: Revisit

$$X(s) = \frac{1}{s(s^2+9)} + \frac{1}{(s^2+9)}$$

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+9)}\right\} + \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}}_{\frac{1}{3}\sin(3t)}$$

$$= \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} d\tau + \frac{1}{3}\sin(3t)$$

$$= \int_0^t \frac{1}{3} \sin(3\tau)\, d\tau + \frac{1}{3}\sin(3t)$$

$$= -\frac{1}{9}\cos(3t) + \frac{1}{9} + \frac{1}{3}\sin(3t) \quad //$$

shortcut if there is an s in denom. of $F(s)$

$$\frac{1}{(s-2)(s^2+9)} \rightarrow ?$$

partial fractions.

Covered 7.2, most of 7.3 (except translation on s axis)