Plan for today:
Finish 7.3
7.4

1. Compute Laplace transform of function involving an exponential or the inverse of a Laplace transform involving a translation
2. Be able to recognize a convolution
3. Be able to use the convolution theorem to compute the Laplace transform of the convolution of two functions
4. Know the formula turning multiplication by t into differentiation in s
5. Know the formula turning division by t into integration in s

Announcements-Reminders
1. Read the Textbook1
2. Synchronous online section (901) takes the final in person on May 4, 7-9 pm. More information on the location will be announced today.
3. Asynchronous online section (OL1) takes the final online on MyLab Math, May 4, 7pm-May 5, 7pm.
4. Quiz grades will be posted by Monday

7.3 (last part) : translation on s axis.

If f is nice enough

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a) \qquad \left(F(s)=\mathcal{L}\{f(t)\}\right)$$

$$\Leftrightarrow \quad \mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$$

Multiplication by exp $\leftrightarrow$ translation in s.

$\hookrightarrow$ Property 14 in Laplace table.

Ex: $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = e^{at}\cdot 1$

Ex: $F(s) = \dfrac{s-1}{(s+1)^3}$

1st way

w/ partial fractions

$$\frac{s-1}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$$

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or

2nd way

$$F(s) = \frac{s+1-2}{(s+1)^3} = \tilde{F}(s+1)$$

$$\tilde{F}(s) = \frac{s-2}{s^3} = \frac{1}{s^2} - \frac{2}{s^3}$$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{\tilde{F}(s+1)\} \underset{a=-1}{\overset{\text{rule}}{=}} e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{2}{s^3}\right\}$$

$$\underset{\text{table.}}{=} e^{-t}\left(t - t^2\right)$$

Convolution

Laplace doesn't play well w/ products of functions.

If c is const. then $\mathcal{L}\{c f(t)\} = c\,\mathcal{L}\{f(t)\}$

but:

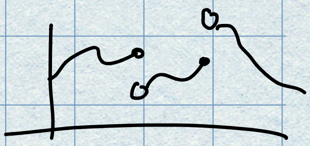
$$\mathcal{L}\{f_1(t)\cdot f_2(t)\} \neq \mathcal{L}\{f_1(t)\}\,\mathcal{L}\{f_2(t)\}$$

$$\left[\begin{array}{l}\text{Take } f_1(t) = 1 = f_2(t) \\ \mathcal{L}\{1\cdot 1\} = \mathcal{L}\{1\} = \frac{1}{s} \\ \mathcal{L}\{1\}\,\mathcal{L}\{1\} = \frac{1}{s}\cdot\frac{1}{s} = \frac{1}{s^2} \;\#\end{array}\right]$$

Conv: an operation between functions which plays well w/ Laplace.

Def'n: f, g piecewise continuous, on $[0, \infty)$

$$f * g(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$



convolution

Conv. is commutative: $f * g = g * f$

$$f * g(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau \overset{u = t-\tau}{=} -\int_t^0 f(t-u)\, g(u)\, du$$

$$= \int_0^t g(u)\, f(t-u)\, du = g * f(t)$$

Convolution Theorem

f, g nice. Then

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$$

table: entry 16

Laplace turns convolution into multiplication.

Ex: Find $\mathcal{L}^{-1}\{F(s)\}$, $F(s) = \dfrac{s}{(s-3)(s^2+1)}$

1st way: Partial fractions

$$\frac{s}{(s-3)(s^2+1)} = \frac{A}{s-3} + \frac{Bs+C}{s^2+1}$$

(Exercise)

2nd way: $\mathcal{L}^{-1}\left\{\frac{s}{(s-3)(s^2+1)}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s-3} \cdot \frac{s}{s^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} * \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$= \left(e^{3t}\right) * \left(\cos(t)\right) \quad *$$

$$= \int_0^t e^{3\tau} \cos(t-\tau)\, d\tau$$

trig identity

$$=$$

$$\cos(t-\tau) = \cos(t)\cos(\tau) + \sin(t)\sin(\tau)$$

$$\int_0^t e^{3\tau} \cos(t)\cos(\tau)\, d\tau + \int_0^t e^{3\tau} \sin(t)\sin(\tau)\, d\tau$$

$$= \cos(t) \underbrace{\int_0^t e^{3\tau}\cos(\tau)\, d\tau}_{?!} + \sin(t) \underbrace{\int_0^t e^{3\tau}\sin(\tau)\, d\tau}_{?!}$$

↪ Alternate way: from ⊛

$$= \cos(t) * e^{3t}$$

$$= \int_0^t \cos(\tau)\, e^{3(t-\tau)}\, d\tau$$

$$= e^{3t} \int_0^t \cos(\tau)\, e^{-3\tau}\, d\tau$$

exercise

$$= \dots = e^{3t}\left(\frac{1}{10} e^{-3t} \sin(t) - 3\cos(t)\, e^{-3t} + 3\right)$$

So: $\mathcal{L}^{-1}\{F(s)\}$

$$= e^{3t}\left(\frac{1}{10} e^{-3t} \sin(t) - 3\cos(t)\, e^{-3t} + 3\right) \quad //$$

Differentiation & integration

Seen: 1. $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$ (table entry 18)

2. $\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s} F(s)$ (not on table)

$\boxed{F(s) = \mathcal{L}\{f(t)\}}$

$$\Leftrightarrow \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t \mathcal{L}^{-1}\{F(s)\}(\tau)\,d\tau$$

Now: Diff/tion, integration in s ↩

multiplication / division in t.

3. $\mathcal{L}\{-tf(t)\} = F'(s)$

$\Leftrightarrow f(t) = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}$ (table entry 19)

4. If $\lim_{t\to 0^+} \frac{f(t)}{t}$ exists & is finite

then: $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma)\,d\sigma$

$\Leftrightarrow \quad f(t) = t\mathcal{L}^{-1}\int_s^\infty F(\sigma)\,d\sigma$ (not on table)

Ex: $\mathcal{L}\{t^2\cos(2t)\}$

By Def'n: $\int_0^\infty e^{-st}\, t^2\cos(2t)\,dt$

IBP not fun

Instead:

$$\mathcal{L}\{t^2\cos(2t)\} = \mathcal{L}\{(-t)(-t)\cos(2t)\}$$

$$= \frac{d}{ds}\underbrace{\mathcal{L}\{(-t)\cos(2t)\}}$$

$$= \frac{d^2}{ds^2}\mathcal{L}\{\cos(2t)\}$$

$$= \frac{d^2}{ds^2}\left(\frac{s}{s^2+4} \right)$$

$$= \dots = \frac{2s(s^2-12)}{(s^2-4)^3} \quad //$$

$$F(s) = \mathcal{L}\{f(t)\}$$

$$f(t) = \cos(2t), \quad F(s) = \underbrace{\mathcal{L}\{\cos(2t)\}(s)}$$

$$\mathcal{L}\{t^2 \cos(2t)\} = \mathcal{L}\{t^2 f(t)\} = \frac{d^2}{ds^2} F(s)$$