Plan for today:
Finish 3.4
3.5

Learning Goals

1. Be able to rewrite a piecewise continuous function as a product (or sum of products) of a continuous function and one or more step functions.
2. Be able to compute the Laplace transform of piecewise continuous functions using the appropriate rule.

Announcements- reminders

1. Synchronous online section (901) takes the final in person on May 4, 7-9 pm in WALC 1055.
2. Asynchronous online section (OL1) takes the final online on MyLab Math, May 4, 7pm-May 5, 7pm.
3. Quiz 6 grades are posted
4. If you are in the OL1 section please take the Mock_Exam assignment (10' mock exam to test respondus). No action needed for 901 Section.

Dif/tion, integration in $s \leftrightarrow$ multiplication / division in t.

3. $\mathcal{L}\{-tf(t)\} = F'(s)$ (table entry 19)

$\Leftrightarrow f(t) = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}$

4. If $\lim_{t\to 0^+} \frac{f(t)}{t}$ exists & is finite $\boxed{F(s) = \mathcal{L}\{f(t)\}}$

then: $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma)\,d\sigma$

$\Leftrightarrow f(t) = t\mathcal{L}^{-1}\int_s^\infty F(\sigma)\,d\sigma$ (not on table)

Ex: $g(t) = \frac{e^t - e^{-t}}{t}$, $\mathcal{L}\{g(t)\}$

$\lim_{t\to 0^+} \frac{e^t - e^{-t}}{t} \overset{\text{L'Hôpital}}{=} \lim_{t\to 0^+} \frac{e^t + e^{-t}}{1} = 2$ ✓

$$\mathcal{L}\left\{\frac{e^{t}-e^{-t}}{t}\right\} = \int_s^\infty \mathcal{L}\{e^{t}-e^{-t}\}(\sigma)\,d\sigma$$

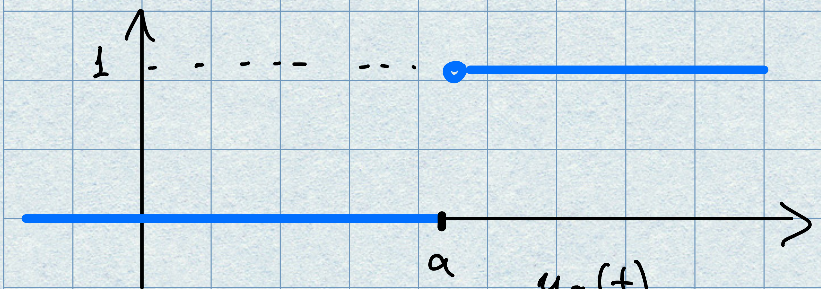
$$= \int_s^\infty \frac{1}{\sigma-1} - \frac{1}{\sigma+1}\,d\sigma$$

$$= \;\dots\; = -\ln\left(\frac{s-1}{s+1}\right) \qquad //$$

$$\lim_{t\to a}\frac{f(t)}{g(t)} \overset{\frac{0}{0}}{=} \lim_{t\to a}\frac{f'(t)}{g'(t)}$$

$$f(a)=g(a)=0.$$

7.5 **Recall:** $u_a(t) = u(t-a) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases}$

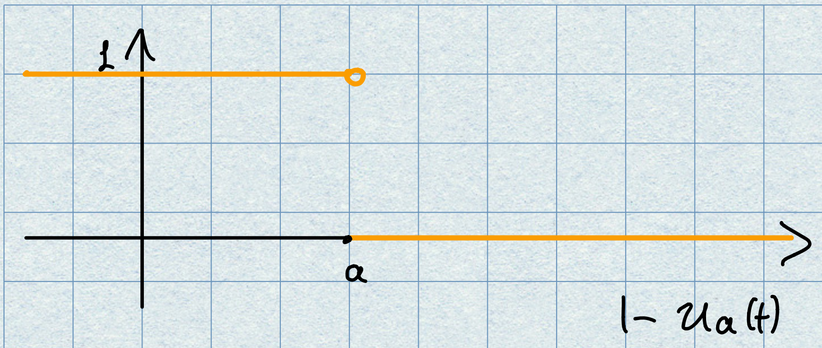


step functions.

$u_a(t)$

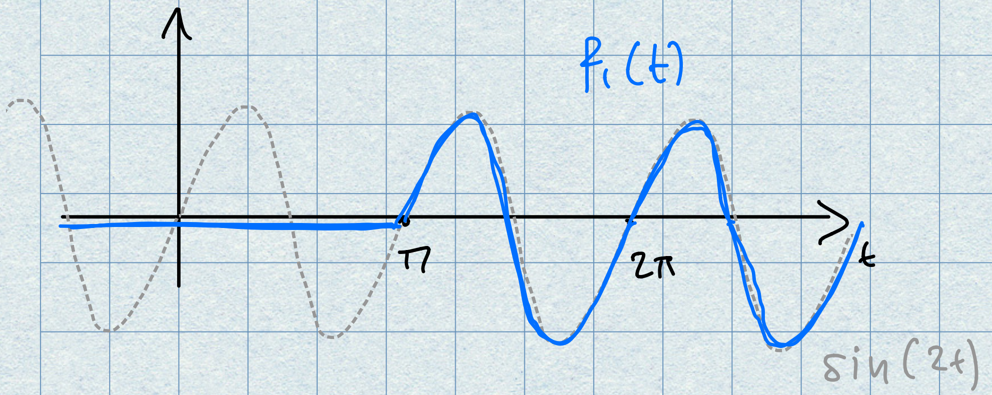
$u(t) = u_0(t)$

Also: $1 - u_a(t) = \begin{cases} 0, & t \geq a \\ 1, & t < a \end{cases}$



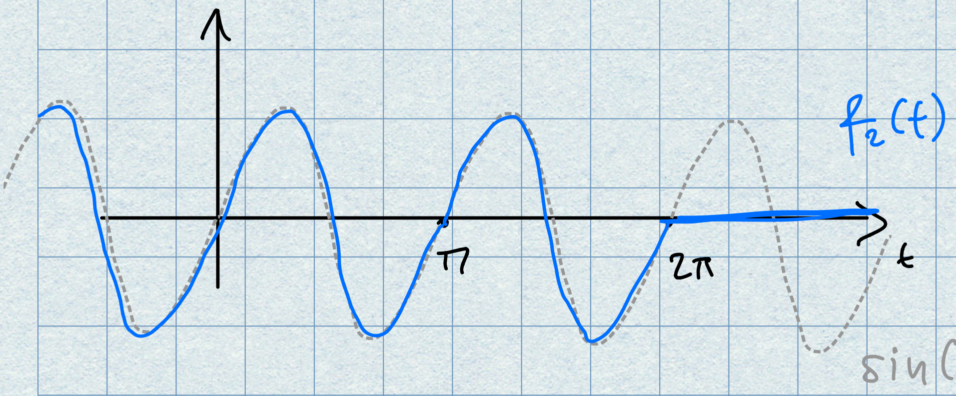
Can use u_a, $1-u_a$ to express signals starting time delay or stopping after certain time.

Ex: $f_1(t) = \begin{cases} \sin(2t) & t \geq \pi \\ 0 & t < \pi \end{cases}$



$$f_1(t) = \underbrace{u(t-\pi)}_{\substack{\shortparallel \\ \begin{cases} 0, & t < \pi \\ 1, & t \geq \pi \end{cases}}} \sin(2t) = u_\pi(t) \sin(2t)$$

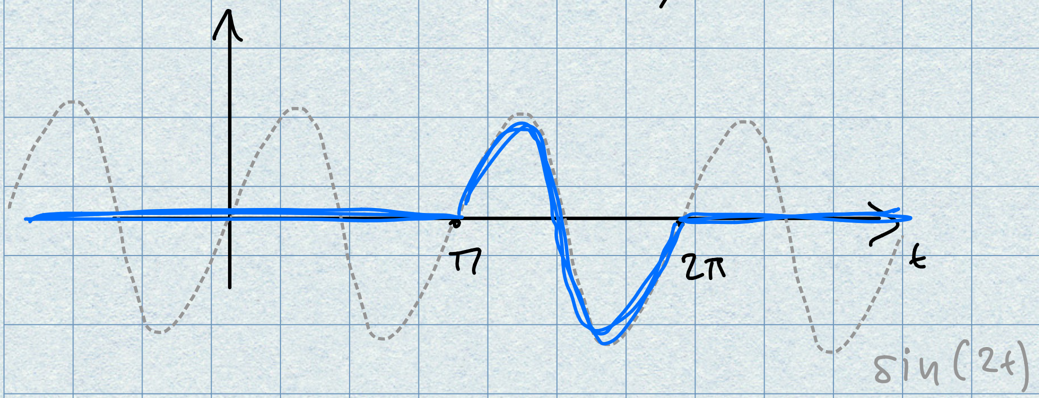
Ex 2: $f_2(t) = \begin{cases} \sin(2t) & t \leq 2\pi \\ 0 & t > 2\pi \end{cases}$



$$f_2(t) = (1 - u_{2\pi}(t))\sin(2t)$$

$$= (1 - u(t-2\pi))\sin(2t)$$

Ex 3: $f_3(t) = \begin{cases} \sin(2t) & \pi \le t \le 2\pi \\ 0, & t < \pi \text{ or } t \ge 2\pi \end{cases}$



$$f_3(t) = u_\pi(t)\left(1 - u_{2\pi}(t)\right)\sin(2t) \quad *$$

$$= u_\pi(t)\, f_2(t)$$

$$= (1 - u_{2\pi}(t))\, f_1(t)$$

Nice observation:

$$\circledast = u_\pi(t)\left(1 - u_{2\pi}(t)\right)\sin(2t)$$

$$= \left(u_\pi(t) - u_\pi(t)\,u_{2\pi}(t)\right)\sin(2t)$$

$$\boxed{f_3(t) = \left(u_\pi(t) - u_{2\pi}(t)\right)\sin(2t)}$$

Check: $u_\pi(t)u_{2\pi}(t) = u_{2\pi}(t)$

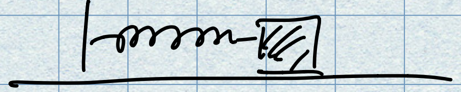
$\circledast$

Application - example. Mass-spring system.

$k = 9$

$m = 1$

$c = 0$



Initially: at rest at equilibrium $(x(0) = x'(0) = 0)$

Force $f(t) = \sin(2t)$ applied at $t = \pi$ until $t = 2\pi$

$$mx'' + cx' + kx = f(t)$$

$$x'' + 9x = f_3(t)$$

Take Laplace tr. $\quad X(s) = \mathcal{L}\{x(t)\}$

$$\left(s^2 X(s) - s\,\underbrace{\cancel{x(0)}}_{0} - \underbrace{\cancel{x'(0)}}_{0}\right) + 9X(s) = \mathcal{L}\{f_3(t)\}$$

$$X(s) = \frac{\mathcal{L}\{f_3(t)\}}{s^2 + 9}$$

$$\Rightarrow \quad x(t) = \mathcal{L}^{-1}\left\{ \frac{\mathcal{L}\{f_3(t)\}}{s^2+9} \right\}$$

Tasks: 1. find $\mathcal{L}\{f_3(t)\}$

2. find $\mathcal{L}^{-1}\left\{ \frac{\mathcal{L}\{f_3(t)\}}{s^2+9} \right\}$

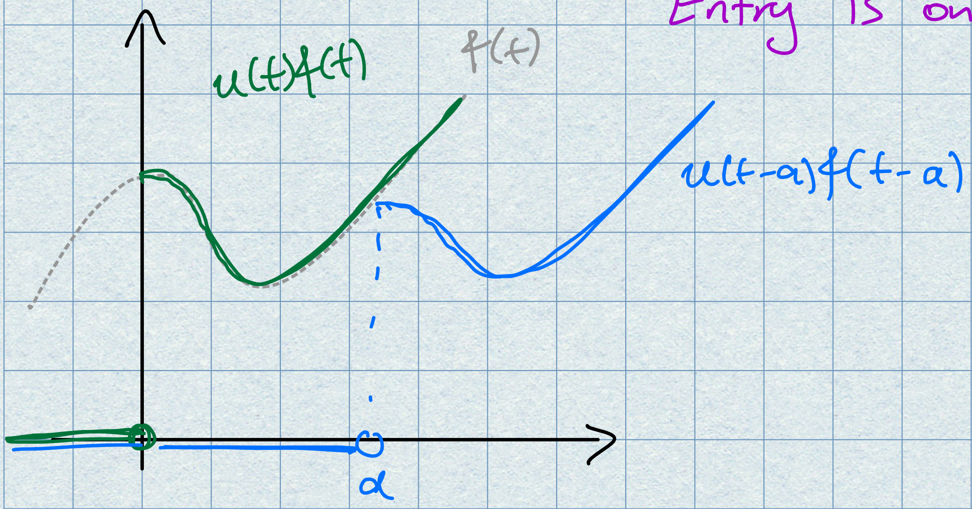
For 1: A new rule:

$$\mathcal{L}\{u(t-a)\, f(t-a)\} = e^{-as}\, \mathcal{L}\{f(t)\}$$

$$\Leftrightarrow \quad \mathcal{L}^{-1}\{e^{-as} F(s)\} = u(t-a)\, f(t-a)$$

"multiplication by exp. in s $\leftrightarrow$ translation in t"

Entry 13 on Laplace table.



Now use rule for $\mathcal{L}\{f_3(t)\}$:

$$(*) \Rightarrow \mathcal{L}\{\sin(2t)\,(u(t-\pi) - u(t-2\pi))\} =$$

$$= \mathcal{L}\{\sin(2t)\, u(t-\pi)\} - \mathcal{L}\{\sin(2t) u(t-2\pi)\}$$

$$= \mathcal{L}\{\sin(2(t-\pi) + \underbrace{2\pi}_{\text{compensates for}})\, u(t-\pi)\}$$

$$- \mathcal{L}\{\sin(2(t-2\pi)+4\pi)\, u(t-2\pi)\}$$

sin(x) period 2π

$$= \mathcal{L}\{\sin(2(t-\pi))\, u(t-\pi)\}$$

$\sin(x+2\pi) = \sin(x)$

$$- \mathcal{L}\{\sin(2(t-2\pi))\, u(t-2\pi)\}$$

Rule

$$= e^{-\pi s}\mathcal{L}\{\sin(2t)\} - e^{-2\pi s}\mathcal{L}\{\sin(2t)\}$$

Table entry # 5

$$= (e^{-\pi s} - e^{-2\pi s})\frac{2}{s^2+4}$$

Computed $\mathcal{L}$ of piecewise cont. fct f_3!

<u>Task</u> 2: Find

$$\mathcal{L}^{-1}\left\{(e^{-\pi s} - e^{-2\pi s})\frac{2}{s^2+4}\frac{1}{s^2+9}\right\}$$

(Wednesday)