Plan for today:
Finish 7.5
7.6

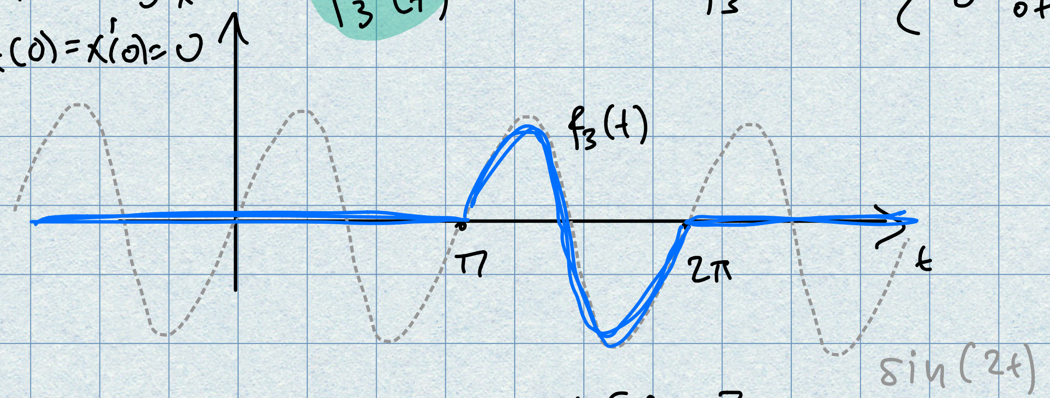
Learning Goals
Be able to solve IVPs involving the delta function

Announcements- reminders
1. Synchronous online section (901) takes the final in person on May 4, 7-9 pm in WALC 1055.
2. Asynchronous online section (OL1) takes the final online on MyLab Math, May 4, 7pm-May 5, 7pm.
3. Send questions for Friday review!

Last time

$x'' + 9x = f_3(t)$ $\qquad f_3(t) = \begin{cases} \sin(2t), & t \in [\pi, 2\pi] \\ 0 & \text{otherwise} \end{cases}$

$x(0) = x'(0) = 0$




Found: $x(t) = \mathcal{L}^{-1}\left\{ \dfrac{\mathcal{L}\{f_3(t)\}}{s^2+9} \right\}$

Tasks: 1. find $\mathcal{L}\{f_3(t)\}$

2. find $\mathcal{L}^{-1}\left\{ \dfrac{\mathcal{L}\{f_3(t)\}}{s^2+9} \right\}$

For 1 found:

$\mathcal{L}\{f_3(t)\} = \left(e^{-\pi s} - e^{-2\pi s}\right) \dfrac{2}{s^2+4}$

Task 2: Find

$$\mathcal{L}^{-1}\left\{\left(e^{-\pi s}-e^{-2\pi s}\right)\frac{2}{s^2+4}\,\frac{1}{s^2+9}\right\}$$

$$=\mathcal{L}^{-1}\left\{e^{-\pi s}\,\frac{2}{s^2+4}\,\frac{1}{s^2+9}\right\}-\mathcal{L}^{-1}\left\{e^{-2\pi s}\,\frac{2}{s^2+4}\,\frac{1}{s^2+9}\right\}$$

Table

$$=u(t-\pi)\,\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\,\frac{1}{s^2+9}\right\}(t-\pi)$$

$$-\,u(t-2\pi)\,\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\,\frac{1}{s^2+9}\right\}(t-2\pi) \quad *$$

$$\boxed{\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\} = u(t-a)\,\mathcal{L}^{-1}\left\{F(s)\right\}(t-a)}$$

Partial fractions $\dfrac{2}{s^2+4}\,\dfrac{1}{s^2+9} = \dfrac{As+B}{s^2+4} + \dfrac{Cs+D}{s^2+9}$

$$= \dots = -\frac{2}{5}\,\frac{1}{s^2+9} + \frac{2}{5}\,\frac{1}{s^2+4}$$

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\,\frac{1}{s^2+9}\right\} \overset{\text{table}}{=} -\frac{2}{15}\sin(3t) + \frac{1}{5}\sin(2t)$$

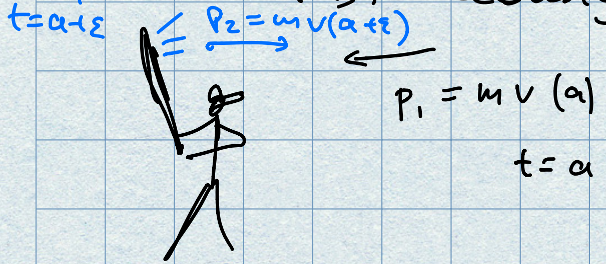
So:

answ

$$(*) = u(t-\pi)\left(-\frac{2}{15}\sin(3(t-\pi)) + \frac{1}{5}\sin(2(t-\pi))\right)$$
$$- u(t-2\pi)\left(-\frac{2}{15}\sin(3(t-2\pi)) + \frac{1}{5}\sin(2(t-2\pi))\right)$$

## 7.6

Goal: Model forces acting almost instaneously




Hard to describe force itself but:

$$\Delta p = p_2 - p_1 = mv(a+\varepsilon) - mv(a)$$
$$\overset{FTC}{=} \int_a^{a+\varepsilon} \frac{d}{dt}(mv)\,dt$$

force.

$$\overset{\text{Newton's law}}{=} \underbrace{\int_a^{a+\varepsilon} f(t)\,dt}$$

impulse of $f$ over interval $[a, a+\varepsilon]$

Quantity $\Delta p$ only depends on integral of force for the time $[a, a+\varepsilon]$

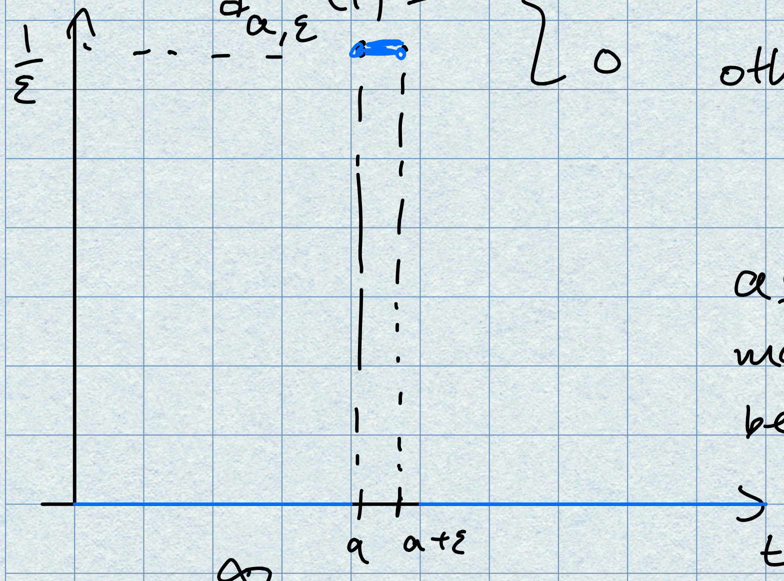
1. set up a simple fct / impulse 1 over

short interval

$$d_{a,\varepsilon}(t) = \begin{cases} \frac{1}{\varepsilon} & a \le t \le a+\varepsilon \\ 0 & \text{otherwise} \end{cases}$$

$\varepsilon$ small

$a > 0$




as $\varepsilon \to$ smaller
max. value of $d_{a,\varepsilon}$
becomes larger.

$$\int_0^\infty d_{a,\varepsilon}(t)\,dt = \int_a^{a+\varepsilon} \frac{1}{\varepsilon}\,dt = \frac{a+\varepsilon}{\varepsilon} - \frac{a}{\varepsilon}$$
$$= 1$$

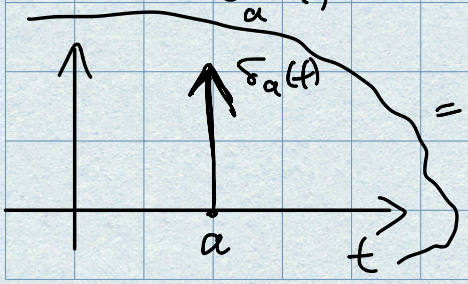
what happens as we send $\varepsilon \to 0$?

<u>Dirac delta function</u> [not a function]

Intuitively think of it as

$$\delta_a(t) = \lim_{\varepsilon \to 0} d_{a,\varepsilon}(t)$$
$$= \begin{cases} \infty & t = a \\ 0 & \text{otherwise} \end{cases}$$

not accurate definition




← in physics

Formally: $\delta_a$ is an operator: eats continuous functions, outputs their value at $a$.

$$g(x) \to \boxed{\;\delta_a\;} \longrightarrow g(a)$$

Notation

$$\int_0^\infty g(t)\,\delta_a(t)\,dt = g(a) \quad \Big]\ \text{definition of } \delta_a$$

not honest integral; notation only.

Motivation for defn:

$$\int_0^\infty g(t)\,d_{a,\varepsilon}(t)\,dt = \int_a^{a+\varepsilon} g(t)\cdot\frac{1}{\varepsilon}\,dt$$

honest integral

$$\overset{\text{FTC}}{=} g(\bar{t}) \quad \text{for some } \bar{t}\in[a,a+\varepsilon]$$

$$\xrightarrow{\varepsilon\to 0} g(a)$$

Ex: $\displaystyle\int_0^\infty 1\,\delta_a(t)\,dt = 1 \qquad a \geq 0$

$$\int_0^\infty e^{-st}\delta_a(t)\,dt = e^{-sa}$$

definition of Laplace of Dirac $\delta$ function

Recall: Lapl: $\int_0^\infty e^{-st} f(t)\,dt$

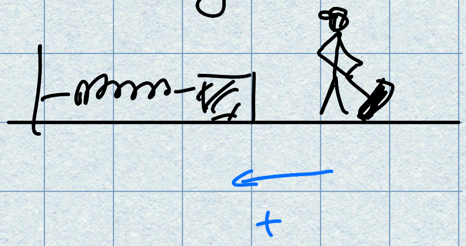
$$\mathcal{L}\{\delta_a(t)\} = e^{-sa}$$ (on table)

$\delta_0 =: \delta$ , $\delta_a(t) = \delta(t-a)$

so $\mathcal{L}\{\delta(t-a)\} = e^{-sa}$

## Solving IVP w/ delta

<u>Ex:</u> Mass-spring system initially at rest




$x(0) = x'(0) = 0$

$m = 1$ mass

$k = 4$ spring const.

no damping

Mass struck w/ hammer at $t = 3$ providing impulse $p = 5$

Displacement:

$$1 \cdot x''(t) + 0 \cdot x'(t) + 4x(t) = 5\,\delta_3(t)$$

impulse

time when impulse happens

Solve:

$$x'' + 4x = 5\,\delta_3(t)$$

Use Laplace!  $X = \mathcal{L}\{x(t)\}$

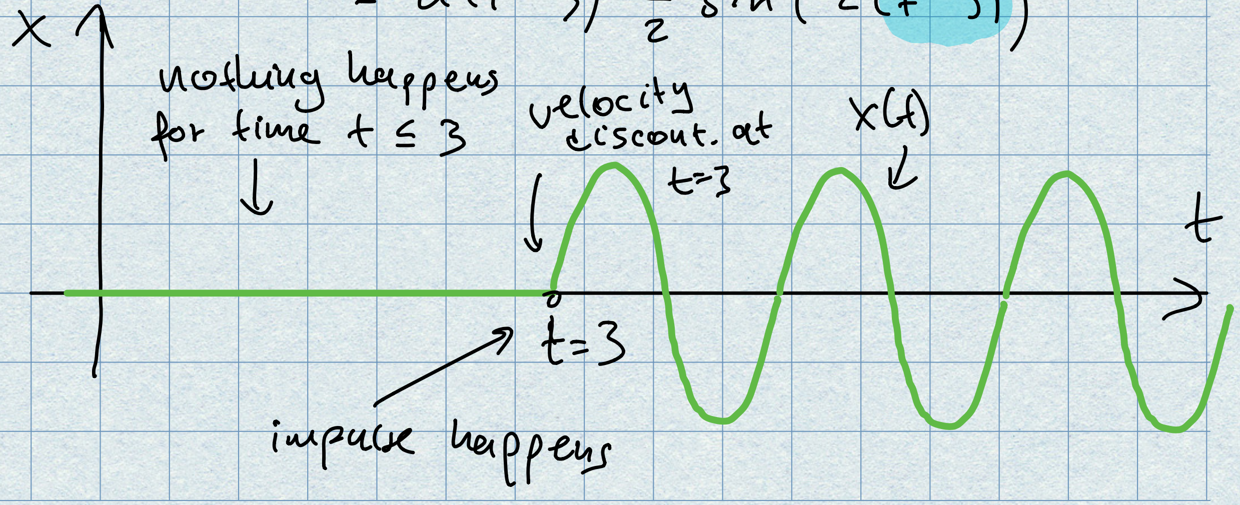
$$s^2 X(s) + 4X(s) = 5e^{-3s}$$

$$\Rightarrow \quad X(s) = \frac{5e^{-3s}}{s^2+4}$$

$$\Rightarrow \quad x(t) = \mathcal{L}^{-1}\left\{\frac{5e^{-3s}}{s^2+4}\right\}$$

$$= u(t-3)\,\mathcal{L}^{-1}\left\{\frac{5}{s^2+4}\right\}(t-3)$$

$$= u(t-3)\,\frac{5}{2}\sin(2(t-3))$$



Duhamel's principle: Friday