Matrices

$m \times n$

$$\underline{\underline{A}} = [a_{ij}] = \underset{\text{rows}}{m}\left\{\overbrace{\begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & & \\ & & & \\ & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}}^{n \text{ columns}}\right.$$

Matrix addition:

$$\underset{m\times n}{\underline{\underline{A}}} + \underset{m\times n}{\underline{\underline{B}}} \rightarrow \text{componentwise addition}$$

Ex:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 7 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 10 & 2 \end{bmatrix}$$

Multiplication by scalar:

$$\underset{\text{scalar}}{c}\, \underline{\underline{A}} \rightarrow \text{multiplication of each comp. of } \underline{\underline{A}} \text{ by } c$$

$$2 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

Note: $\underline{\underline{A}} + \underline{\underline{B}} = \underline{\underline{B}} + \underline{\underline{A}}$

Transpose:

$\underline{\underline{A}}^T$: if A is $m \times n$ then A^T is $n \times m$ & its i-th row is the i-th column of A.

Ex:

$$\underline{\underline{A}} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad \underline{\underline{A}}^T = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & 6 \end{bmatrix}$$

3×2 $\qquad$ 2×3

Column vector: $n \times 1$ matrix.

Ex: $\underline{\underline{a}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ 3×1 column vector.

Row vector: $1 \times n$ matrix

$\underline{\underline{b}} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ 1×3 row vector.

Convenient to write

$$\underline{\underline{A}} = \begin{bmatrix} \underline{\underline{a}}_1 & \underline{\underline{a}}_2 & \cdots & \underline{\underline{a}}_n \end{bmatrix}$$

$m \times n$ matrix.

$m \times 1$ column vectors

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \end{bmatrix} = \begin{bmatrix} \underline{\underline{a}}_1 & \underline{\underline{a}}_2 & \underline{\underline{a}}_3 \end{bmatrix}$$

$\underline{\underline{a}}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\underline{\underline{a}}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\underline{\underline{a}}_3 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

Dot product (scalar product)

$$\underset{=}{a} \cdot \underset{=}{b} = \sum_{j=1}^{m} a_j \, b_j$$

entries of $\underset{=}{b}$

$1 \times m$ row vector

$m \times 1$ col. vector

entries of $\underset{=}{a}$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = -1 + 4 + 12 = 15$$

Product of Matrices

$$\underset{=}{A} = \begin{bmatrix} \underset{=}{a}_1 \\ \vdots \\ \underset{=}{a}_m \end{bmatrix} \qquad \underset{=}{B} = \begin{bmatrix} \underset{=}{b}_1 & \cdots & \underset{=}{b}_n \end{bmatrix}$$

$1 \times p$ row vectors

b_j $p \times n$ column vectors

$$\underset{m \times p}{\underset{=}{A}} \; \underset{p \times n}{\underset{=}{B}} = \begin{bmatrix} \underset{=}{a}_1 \cdot \underset{=}{b}_1 & & \underset{=}{a}_1 \cdot \underset{=}{b}_m \\ \vdots & & \\ \underset{=}{a}_n \cdot \underset{=}{b}_1 & & \underset{=}{a}_n \cdot \underset{=}{b}_m \end{bmatrix}$$

$m \times n$ matrix.

$$\underset{2 \times 3}{\begin{bmatrix} 2 & 1 & -4 \\ 4 & -2 & 1 \end{bmatrix}} \underset{3 \times 2}{\begin{bmatrix} 1 & 2 \\ -1 & -3 \\ 4 & 0 \end{bmatrix}} =$$

$$= \begin{bmatrix} 2\cdot 1 + 1\cdot(-1) + (-4)\cdot 4 & 2\cdot 2 + 1\cdot(-3) + (-4)\cdot 0 \\ 4\cdot 1 + (-2)\cdot(-1) + 1\cdot 4 & 4\cdot 2 + (-2)(-3) + 1\cdot 0 \end{bmatrix}$$

2×2

<u>Note:</u> $\underset{m\times p}{\underline{\underline{A}}} \cdot \underset{p\times n}{\underline{\underline{B}}} \neq \underbrace{\underline{\underline{B}} \cdot \underline{\underline{A}}}$

$m \times n$

not generally defined if $\underline{\underline{A}} \to m \times p$, $\underline{\underline{B}} \to p \times n$

<u>Ex:</u> $\underline{\underline{A}} \cdot \underline{\underline{B}} = \underline{\underline{0}} \not\Rightarrow \underline{\underline{A}} = \underline{\underline{0}}$ or $\underline{\underline{B}} = \underline{\underline{0}}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

<u>Identity Matrix:</u>

$$\underline{\underline{I}} = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & \ddots & \\ 0 & 0 & & 1 \end{bmatrix}$$

$$\underset{n\times n}{\underline{\underline{A}}} \cdot \underset{n\times n}{I} = \underline{\underline{A}} = \underset{n\times n}{\underline{\underline{A}}} \cdot \underset{n\times n}{\underline{\underline{I}}}$$

Given $\underset{n\times n}{\underline{\underline{A}}}$, is there $\underset{n\times n}{\underline{\underline{B}}}$ so that $\underline{\underline{A}} \cdot \underline{\underline{B}} = \mathrm{Id} = \underline{\underline{B}} \cdot \underline{\underline{A}}$?

If $\det \underline{\underline{A}} \neq 0$ then yes, this matrix $\underline{\underline{B}}$ is denoted by $\underline{\underline{A}}^{-1}$!

Find inverse of 2×2 matrix:

$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \det \underline{\underline{A}} \neq 0$$

$$\underline{\underline{A}}^{-1} = \frac{1}{\det \underline{\underline{A}}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Check that this works:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \frac{1}{\det \underline{\underline{A}}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$= \frac{1}{\det \underline{\underline{A}}} \begin{bmatrix} \overbrace{a_{11} a_{22} - a_{12} a_{21}}^{\det A} & \cancel{a_{11}(-a_{12}) + a_{12}(a_{11})} \to 0 \\ \cancel{a_{21} a_{22} + a_{22}(-a_{21})} \to 0 & \underbrace{a_{21}(-a_{12}) + a_{22} a_{11}}_{\det A} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Check: $\frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = I.$