Today:
Finish 7.6
Review

Duhamel's principle

IVP $\begin{cases} a x'' + b x' + c x = \underbrace{f(t)}_{} \\ x(0) = x'(0) = 0 \end{cases}$ external force e.g. mass spring system

Take Laplace: bec. $x(0) = x'(0) = 0$

$$a s^2 X(s) + b s X(s) + c X(s) = \mathcal{L}\{f(t)\}$$

$X(s) = \mathcal{L}\{x(t)\}$

$$X(s) = \underbrace{\frac{1}{a s^2 + bs + c}}_{\text{transfer function } W(s)} \cdot \mathcal{L}\{f(t)\}$$

$$w(t) = \mathcal{L}^{-1}(W(s))$$

By conv. theorem:

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\{X(s)\} \\ &= \mathcal{L}^{-1}\{W(s)\,\mathcal{L}\{f(t)\}\} \\ &= w(t) * f(t) \\ &= \underbrace{\int_0^t w(\tau) f(t-\tau)\, d\tau}_{\text{Duhamel's formula.}} \end{aligned}$$

w: does not depend on f, only on the system, not on external force f(t), Duhamel's f. gives gives response of system to any input function

So: given mass-spring s., if we know $w(t)$ we can predict response to any external force

Bernoulli eqns (Ch 1, 1.6)

At end of textbook there is index listing where each topic is covered

$$y' + p(x)y = q(x)y^n$$

$$v = y^{1-n}$$

E.g.: $y' + xy = x^2y^3$

$$v = y^{-2} \Rightarrow y = v^{-\frac{1}{2}}$$

$$y' = -\frac{1}{2} v^{-\frac{3}{2}} v'$$

so: $-\frac{1}{2} v^{-\frac{3}{2}} v' + x v^{-\frac{1}{2}} = x^2 v^{-\frac{3}{2}}$

$$-\frac{1}{2} v' + x \underbrace{v^{\frac{3}{2}} v^{-\frac{1}{2}}} = x^2$$

$$-\frac{1}{2} v' + x v = x^2$$

$$v' + 2xv = 2x^2$$

Linear eqn of 1st order.

$$p = e^{\int 2x\,dx} \quad \text{(integrating factor)}$$
$$= e^{x^2}$$

$$(pv)' = 2x^2 e^{x^2}$$

$$\Rightarrow pv = \int 2x^2 e^{x^2}\,dx$$
$$= \int x \frac{d}{dx}\left(e^{x^2}\right) dx$$
$$= x e^{x^2} - \underbrace{\int e^{x^2} dx}_{\text{can't integrate (bad choice of example)}} \quad //$$

Fall 2019 # 6 is also a Bernoulli eq'n.

Fall 2017 # 11.

Variation of param:

$$y'' + p(x) y' + q(x) y = f(x)$$

y_1, y_2 are 2 lin. indep. sols

Know:

$$-y_1(x) \int \frac{y_2(x)\, f(x)}{W(y_1,y_2)(x)}\,dx + y_2(x) \int \frac{y_1(x)\, f(x)}{W(y_1,y_2)(x)}\,dx$$

$$t^2 y'' - 4t\,y' + 6y = t^3$$

$\boxed{\text{divide by } t^2}$ $\quad y'' - \frac{4}{t} y' + \frac{6}{t^2} y = t$

$$y_1 = t^2, \quad y_2 = t^3$$

$$W(t^2, t^3) = \begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix} = 3t^4 - 2t^4 = t^4$$

$$y_p = -t^2 \int \frac{t^3 \, t}{t^4} dt + t^3 \int \frac{t^2 \, t}{t^4} dt$$

$$= -t^2 \cdot (t + c_1) + t^3 (\ln t + c_2)$$

$$= -t^3 + c_1 t^2 + c_2 t^3 + t^3 \ln t$$

$$= \underbrace{(c_2 - 1)}_{A} t^3 + \underbrace{c_1}_{B} t^2 + t^3 \ln t$$

Answ.: A.

As a rule: variation of param. if

$$y'' + \underbrace{p(x)}_{\uparrow} y' + \underbrace{q(x)}_{\uparrow} y = f(x)$$

non-const. coef.

Might be given one sol'n of
$$y'' + p(x)y' + q(x)y = 0$$
and asked to find a second w/ reduction of order.

Won't be asked to find 2 sol's of a non-const. coef. linear eq'n of 2nd order unless it's Euler eq'n:
$$a t^2 y'' + b t y' + c y = 0$$
Substitution: $t = e^x$

$\Rightarrow x = \ln(t)$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{1}{t}\frac{dy}{dx} = \frac{1}{e^x}\frac{dy}{dx}$$
$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{1}{e^x}\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{e^x}\frac{dy}{dx}\right)\frac{dx}{dt}$$
$$= -\frac{1}{e^{2x}}\frac{dy}{dx} + \frac{1}{e^{2x}}\frac{d^2y}{dx^2}$$
$$\underbrace{a\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right) + b\frac{dy}{dx} + cy = 0}$$

const. coef. linear eq'n, can solve using charact. eq'n.

14 F 2017

$$f(t) = \begin{cases} t, & 0 \le t < 2 \\ 2, & t \geq 2 \end{cases}$$

Lapl?

Use step fct to transform

$$f_1(t) = \begin{cases} t, & 0 \le t < 2 \\ 0, & t \geq 2 \end{cases}$$

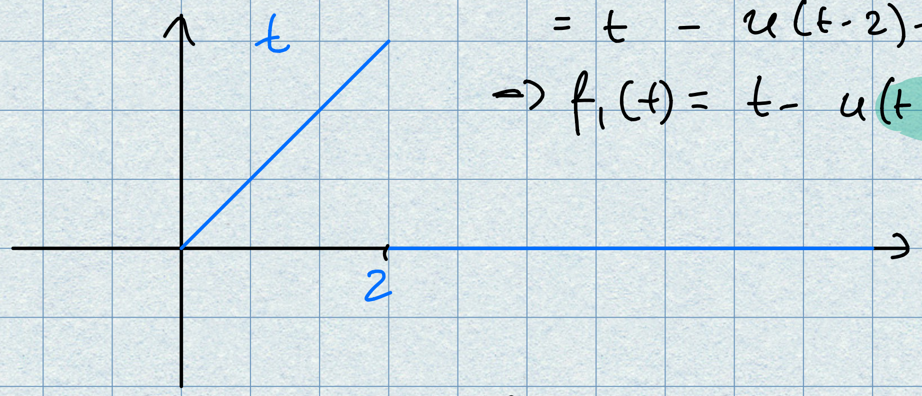
$$f_2(t) = \begin{cases} 0, & 0 \le t < 2 \\ 2, & t \geq 2 \end{cases}$$

$$f(t) = f_1(t) + f_2(t)$$

$$f_1(t) = \left(1 - u_2(t)\right) t \qquad t \geq 0$$

$$= t - u(t-2)t$$

$$\Rightarrow f_1(t) = t - u(t-2)\left((t-2)+2\right)$$



$$f_2(t) = 2\, u_2(t)$$

(8th entry 1st column)

Use table to compute

$$\mathcal{L}\{f\} = \mathcal{L}\{f_1\} + \mathcal{L}\{f_2\}$$