Reminder: Quiz 2 (Graded) will become available tomorrow 9.30AM

MA 266, Spring 2021
Recap of §1.1-2.5, Recipe Book, and Problems

02/21/2021

Non-comprehensive list of topics covered

Chapter 1

- §1.1: What is a differential equation? What is a solution for a differential equation? Know how to check whether a given function is a solution to a differential equation. Order of a differential equation, difference between ordinary versus partial differential equation. What is an Initial Value Problem for a first order differential equation?

- §1.2: General Solutions, Particular Solutions. Be able to solve differential equations of the form $\frac{dy}{dx} = f(x)$. Be able to convert between physical units.

- §1.3: Slope fields, solution curves. Be able to check whether the Theorem on existence and uniqueness of solutions applies to a given differential equation (Theorem 1 in 1.3).

- §1.4: What is a separable differential equation? Be able to identify one when you see it in Nature ("Nature" means exams) and solve it. Implicit solutions of differential equations, singular solutions. Population growth, Newton's Law of Cooling, radioactive decay.

- §1.5: What is a linear 1st order ODE? Be able to identify linear 1st order ODEs in Nature and solve them using an integrating factor. Existence and uniqueness theorem for 1st order linear ODEs. Know the difference in the assumption and the conclusions between this theorem and the theorem on existence and uniqueness from Section 1.3. Mixture problems.

- §1.6: Know the process of using a substitution to solve a differential equation. Be able to identify homogeneous equations (in the sense $\frac{dy}{dx} = f(y/x)$) and Bernoulli equations and solve them with the appropriate substitutions. Be able to check whether a differential equation in differential form is exact and solve it.

Be able to solve reducible 2nd order differential equations with the appropriate substitutions. Be able to use non-standard substitutions if needed, in the spirit of Example 6.

Chapter 2

- §2.1: Be able to set up a differential equation that describes population growth. Be able to find the parameters for the logistic equation, identify the critical points and the carrying capacity.

- §2.2: Be able to identify an autonomous equation. Be able to drawn and interpret a phase diagram. Stable and unstable critical points and equilibrium solutions. Logistic equation with harvesting, bifurcation diagrams.

- §2.3: Be able to set up and solve problems where the motion of a body is affected by air or fluid resistance.

- §2.4-2.5. Be able to apply Euler's method and the improved Euler's method to approximate solutions of a given equation. You should be able to carry out a few steps of each algorithm by hand. Be able to compute the error in the approximation between a known exact solution and the Euler/Improved Euler approximation.

Recipe book

Below is a summary of the most important entries in our ODE recipe book covered in the sections above.
Please note: the examples and non-examples are not necessarily made to be easy to solve.

1. **Antiderivatives:**
$$\frac{dy}{dx} = f(x) \qquad \text{(no } y \text{ on the right hand side).} \tag{1}$$
Solve by integrating both sides with respect to x.
   - Examples: $\frac{dy}{dx} = \cos(x)$, $\frac{dx}{dt} = t^2$.
   - Non-examples: $y' = yx$, $y' = x + y$.

2. **Separable:**
$$\frac{dy}{dx} = f(x)g(y)$$
Solve by separating variables, i.e. writing $\frac{dy}{g(y)} = f(x)dx$, and integrating both sides.
   - Examples: $y' = yx$, $\dfrac{dx}{dt} = tx + t\sin(x)$
   - Non-examples: $y' = x + y$, $y' = xy + y^3$

3. **Linear:**
$$\frac{dy}{dt} + P(x)y = Q(x)$$
Solve by finding an integrating factor $\rho(x) = e^{\int P(x)dx}$ and reducing to
$$\frac{d}{dx}\left(\rho(x)y(x)\right) = \rho(x)Q(x),$$
which is of the form (1).
   - Examples: $y' + xy = \cos(x)$, $\dfrac{dx}{dt} = x\sin(t) + \dfrac{1}{t}$
   - Non-examples: $y' + xy^2 = \cos(x)$, $y' + xy = \cos(x)y^2$, $y' + xy = \cos(y)$.

4. Of the form
$$\frac{dy}{dx} = F(ax + by + c).$$
Use the substitution $v = ax + by + c$ to reduce to $\dfrac{dv}{dx} = a + bF(v)$, which is separable.
Example: $\dfrac{dy}{dx} = \frac{x+y+2}{x+y-1}$.

5. **Homogeneous:**
$$\frac{dy}{dx} = F(y/x).$$
Quick way to identify: replace x by λx and y by λy in the equation and see if the λ cancel out.
Solve by setting $v = y/x$ and reducing to
$$x\frac{dv}{dx} + v = F(v),$$
which is separable.

   - Example: $(y^2 + x^2)\dfrac{dy}{dx} = \dfrac{x^3 + xy^2}{x + y}$
   - Non-example: $(y^2 + x^2)\dfrac{dy}{dx} = x^3 + xy^2$

6. **Bernoulli:**
$$\frac{dy}{dt} + P(x)y = Q(x)y^n.$$
Solve by setting $v = y^{1-n}$ and reducing to a linear equation.

   - Examples: $y' + x^2 y = \cos(x)y^3$, $y' = xy$, $y' = xy + \cos(x)$, $y' = \sin(x)y^3$
   - Non-example: $y' + xy^2 = \sin(x)y^3$

   Note: some of the examples above are linear or separable so it is easier to treat them as such.

7. **Exact:**
$$M(x,y)dx + N(x,y)dy = 0 \text{ with } \frac{dM}{dy} = \frac{dN}{dx} \text{ on a rectangle.}$$
Solve by setting $\partial_x F(x,y) = M(x,y)$, integrating to find
$$F(x,y) = \int M(x,y)dx + g(y), \tag{2}$$
and substituting (6) into $\partial_y F(x,y) = N(x,y)$ to find $g(y)$ and with it F.

   - Example: $2xydx + x^2 dy = 0$
   - Non-example: $2xydx - x^2 dy = 0$

8. **Reducible 2nd Order:** One of the following types:

(a) y missing:

$$\frac{d^2y}{dx^2} = F\left(\frac{dy}{dx}, x\right)$$

Solve by setting $v = \frac{dy}{dx}$. This reduces to

$$\frac{dv}{dx} = F(v, x). \tag{3}$$

Solve (3), recover $v = \frac{dy}{dx}$, and solve one more differential equation to recover y.

- Example: $y'' + 2y' = x$
- Non-example: $y'' + 2y' + 3xy = 0$

(b) x missing:

$$\frac{d^2y}{dx^2} = F\left(\frac{dy}{dx}, y\right) \tag{4}$$

Solve by regarding y as the independent variable and setting $v = \frac{dy}{dx}$, which reduces (4) to

$$v\frac{dv}{dy} = F\left(v, y\right). \tag{5}$$

Solve (5), recover $v = \frac{dy}{dx}$, and solve one more differential equation to recover y.

- Example: $y'' + 2y' = 9y$
- Non-example: $y'' + 2y + 3xy = 0$

Note: The resulting equations (3), (5) may or may not be easy to solve, but at least they are of 1st order.

$$y'' + 2y' = 9y \qquad (v, y)$$

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = v\frac{dv}{dy}$$

$$v\frac{dv}{dy} + 2v = 9y$$

$$\frac{dv}{dy} = 9\frac{y}{v} - 2 \quad \text{homogeneous.}$$

Problems

You will probably find some of the problems in this list challenging. Do as many as you can and we will discuss them in class.

1. A population of Pallas cats relies on chance encounters of males and females for reproductive purposes. Thus the birth rate β (births per unit of time per unit of population) for this population is an **increasing** linear function of the population itself. That is, $\beta(P) = \beta_0 + \alpha P$, where β_0, α are positive constants.

   Moreover, the population is recovering from a disease and their death rate δ (deaths per unit of time per unit of population) is decreasing with time: it is of the form $\delta(t) = \delta_0 + \hat{\alpha} e^{-t}$, where t is measured in months and $\delta_0, \hat{\alpha}$ are positive constants.

   a. Find the differential equation that describes the population growth of the Pallas cats

   b. which of the following characterizations apply to it?

   - Separable ✗
   - Homogeneous ✗
   - Bernoulli ✓
   - Logistic ✗
   - Linear ✗ (bec of P^2)
   - Autonomous ✗

$\frac{\text{deaths}}{(\text{unit of time})(\text{unit of pop})}$

$$\Delta P \sim (\beta - \delta) P \Delta t$$

$\frac{\text{births}}{(\text{unit of time})(\text{unit of pop})}$

$$\frac{dP}{dt} = (\beta - \delta) P$$
$$= (\beta_0 + \hat{\alpha} P - \delta_0 - \hat{\alpha} e^{-t}) P$$
$$= (\beta_0 - \delta_0 - \hat{\alpha} e^{-t}) \underset{\sim}{P} + \hat{\alpha} \underset{\sim}{P^2}$$

dep. variable

dep. var. squared

logistic model:

$\beta = \beta_0 - \beta_1 P$

$\delta = \delta_0$

$$\frac{dP}{dt} = (\beta_0 - \beta_1 P - \delta_0) P$$

2. You are given the following differential equations:

$$\frac{dy}{dt} = -e^t y + \underbrace{\frac{1}{t+1}}_{f(t,y)}$$

i. $\dfrac{dy}{dt} = \sqrt{y}$ ii. $\dfrac{dy}{dt} + e^t y = \cos(t)$ iii. $\dfrac{dy}{dt} = e^y$ iv. $\dfrac{dy}{dt} + e^t y = \dfrac{1}{t+1}$

a. For which one(s) of them can we guarantee that there exists a unique continuous solution $y(t)$ which satisfies $y(0) = 0$ and is defined for all t in some small interval about $t = 0$?

b. For which one(s) of them can we guarantee that there exists a unique continuous solution $y(t)$ which satisfies $y(0) = 0$ and is defined for all $t \in \mathbb{R}$?

The two theorems:

1. "General" Ex & Uniqueness (from §1.3)

$$\begin{cases} \dfrac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

$$\frac{dy}{dt} + \overbrace{e^t}^{P(t)} y = \overbrace{\frac{1}{t+1}}^{Q(t)}$$

f, $D_y f$ cont. near (x_0, y_0) $\Rightarrow$ there is unique sol'n in some interval containing x_0

2. Ex. & Uniqueness for linear eq's (from §1.5)

$$\begin{cases} \dfrac{dy}{dx} + P(x)y = Q(x) \\ y(x_0) = y_0 \end{cases}$$

If P, Q cont. on an interval which contains x_0 then IVP has a unique sol'n in the entire interval

So for a: ii, iii, iv. : In all these cases $\frac{dy}{dt} = f(t,y)$ w/ $f, \partial_y f$ cont. near $(0,0)$. So ex. & uniqueness follows from first thm.

for b: ii : Only ii, iv are linear. Only ii has P, Q cont. in all of $\mathbb{R}$

3. You are given the following equation in differential form:

$$(x^3 + \underset{M}{\underset{\shortparallel}{yx^{-1}}})dx + (y^2 + \underset{N}{\underset{\shortparallel}{\ln(x)}})dy = 0. \tag{6}$$

a. Show that (6) is exact.

b. Solve (6) with the initial condition $y(1) = 2$.

c. Suppose that you want to use an Euler's method calculator such as

https://www.emathhelp.net/calculators/differential-equations/euler-method-calculator/

to apply Euler's method with step size 0.1 to approximate $y(2)$, where $y(x)$ is the solution of (6) with the initial condition $y(1) = 2$. What should you enter in the various fields in the link to obtain the desired approximation? What is the value of the approximation?

d. Use your answer from part b. and a computer algebra system to find $y(2)$ numerically. Compare your answer with the approximation in part c.

a) $\partial_y M \overset{?}{=} \partial_x N$

$\partial_y M = x^{-1}$, $\partial_x N = x^{-1}$ so exact.

b) Let f be s.t.

$\partial_x f = x^3 + yx^{-1}$, $\partial_y f = y^2 + \ln x$

$\partial_x f = x^3 + yx^{-1}$

$\Rightarrow f(x,y) = \frac{x^4}{4} + y\ln x + g(y)$

$\partial_y f = y^2 + \ln x$

$\Rightarrow \ln x + g'(y) = y^2 + \ln x \Rightarrow g(y) = \frac{y^3}{3} + C$

So general sol'n is given by

$f(x,y) = \frac{x^4}{4} + y\ln x + \frac{y^3}{3} + C = 0$

Want $y(1) = 2$ so

$$\frac{1}{4} + 2\ln 1 + \frac{8}{3} + c = 0$$

$$\Rightarrow \quad c = -\frac{35}{12}$$

Sol: $\boxed{\frac{x^4}{4} + y\ln x + \frac{y^3}{3} - \frac{35}{12} = 0}$ (*)

c. For Euler: write $\frac{dy}{dx} = \underbrace{-\frac{x^3 + yx^{-1}}{y^2 + \ln(x)}}_{\text{this will be } f.}$

Step size: 0.1

$y(1) = 2$

Find $y(2) = -0.573$

d. In (*) plug in $x = 2$, use a CAS to find $y \sim -1.0329$

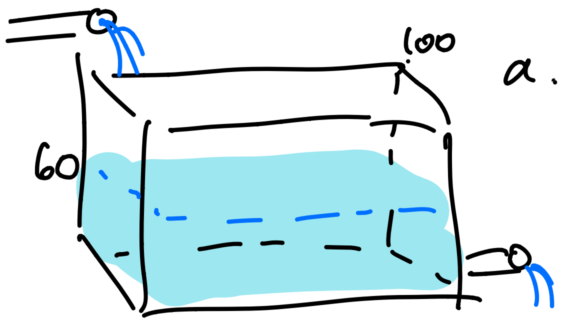
So Euler's method is not very accurate.

W/ step size $h = 0.01$ it gives -1.0129, which is closer.

4. A container with capacity $100lt$ contains $60lt$ of a solution of ethyl alcohol with concentration 50% and a population of bacteria. At time $t = 0$ a solution of ethyl alcohol with concentration $\left(80+\frac{60}{60+t}\right)$% starts flowing into the container at a rate of $2lt/min$. At the same time the well mixed solution starts flowing out of the tank at a constant rate of $1lt/min$. The bacteria in the tank will die once the concentration of the solution in the tank becomes 70%.

   a. Find the concentration of ethyl alcohol in the tank after t min

   b. Do the bacteria die before the tank overflows?



a. $r_{in} = 2 \frac{lt}{min}$

$r_{out} : 1 \frac{lt}{min}$

$x(t) \rightarrow$ amount of e. alcohol in lt at time t.

$$\frac{dx}{dt} = r_{in} \cdot c_{in} - r_{out} \cdot c_{out}$$

$$= 2 \frac{1}{100}\left(80 + \frac{60}{60+t}\right) - 1 \frac{x(t)}{Vol(t)}$$

r_{in}

$$Vol(t) = \underset{\substack{\uparrow \\ \text{initial volume}}}{60} + 2t - \underset{\substack{\uparrow \\ r_{out}}}{1} \cdot t$$

$$= \frac{2}{100}\left(80 + \frac{60}{60+t}\right) - 1 \frac{x(t)}{60+t}$$

$$\frac{dx}{dt} + \frac{x}{60+t} = \frac{1}{50}\left(80 + \frac{60}{60+t}\right)$$

linear!

solve w/ integrating factor

Integrating factor $\rho = e^{\int \frac{1}{60+t} dt} = e^{\ln(60+t)}$
$= 60+t$

$\Rightarrow \frac{d}{dt}\left((60+t)\, x(t)\right) = \int 1.6 \cdot (60+t) + 1.2\, dt$

$= 97.2t + 0.8t^2 + C$

So $x(t) = \dfrac{97.2t + 0.8t^2 + C}{60+t}$

$x(0) = 0.5 \cdot 60 = 30$ lt so $C = 1800$

Conc. at time t:

$$c(t) = \frac{97.2t + 0.8t^2 + 1800}{(60+t)^2}$$

b. Bacteria die when

$c(t) = 0.7 \Rightarrow$

$1800 + 97.2t + 0.8t^2 = 0.7\left(t^2 + 120t + 3600\right)$

$\Rightarrow 0.1t^2 + 13.2t - 720 = 0$

$\Rightarrow t \sim 41.5$ min

so they don't die since overflow happens at $t = 40$.

5. You are given the following differential equation:

$$-x^2 \sin(y) \frac{dy}{dx} = x\cos(y) + \cos^2(y) + x^2 \tag{7}$$

a. Use an appropriate substitution to reduce (7) to a homogeneous equation. Hint: try $v = \cos(y)$.

b. By solving the homogeneous equation in part a) find an explicit general solution for (7).

c. Find a particular solution that is defined in an open interval containing $x = 1$ (the open interval can be as small as you please).

a. $v(x) = \cos(y(x))$

$$\frac{dv}{dx} = -\sin(y)\frac{dy}{dx}$$ Substitute

$$x^2 \frac{dv}{dx} = xv + v^2 + x^2$$

b. Is this homogeneous? replace $v \to \lambda v$, $x \to \lambda x$ see if λ cancel.

$$\frac{dv}{dx} = \frac{v}{x} + \left(\frac{v}{x}\right)^2 + 1$$

$$= F\left(\frac{v}{x}\right), \quad F(z) = z + z^2 + 1$$

homogeneous.

$u = \frac{v}{x} \Rightarrow v = ux$ $\qquad \frac{dv}{dx} = u + \frac{du}{dx}x$

$$\cancel{u} + \frac{du}{dx}x = \cancel{u} + u^2 + 1$$

$$\Rightarrow \int \frac{du}{u^2+1} = \int \frac{dx}{x} \Rightarrow \arctan(u) = \ln|x| + C$$

$$\Rightarrow \arctan\left(\frac{v}{x}\right) = \ln|x| + C$$

$$\Rightarrow \arctan\left(\frac{\cos(y)}{x}\right) = \ln|x| + C$$

$$\Rightarrow \cos(y) = x\tan(\ln|x| + C)$$

$$\Rightarrow y = \arccos(x\tan(\ln|x| + C))$$

c. $y = \arccos(\underbrace{x\tan(\ln|x| + c)})$

must be in $[-1,1]$ to make sense of arccos

taking $c = 0$ would work, since at $x = 1$ we have $x\tan(\ln|x|)\big|_{x=1} = 0$

Reducible

$y'' + 2y' = x$ $\qquad v = \frac{dy}{dx}$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dv}{dx}$$

$\frac{dv}{dx} + 2v = x$ *linear!*

$$p(x) = e^{\int 2\, dx} = e^{2x}$$

$$e^{2x}\frac{dv}{dx} + 2e^{2x}v = xe^{2x}$$

$$\frac{d}{dx}\left(e^{2x}v\right) = xe^{2x}$$

$$e^{2x}v = \int xe^{2x}\,dx$$

$$e^{2x}v = \int x \frac{d}{dx}\left(\frac{e^{2x}}{2}\right)dx$$

$$= x\frac{e^{2x}}{2} - \int \frac{e^{2x}}{2}\,dx$$

$$= x\frac{e^{2x}}{2} - \frac{e^{2x}}{4} + C$$

this gives v.

Solve $\frac{dy}{dx} = e^{-2x}\left(x\frac{e^{2x}}{2} - \frac{e^{2x}}{4} + C\right)$ to find y.