Review problems for Chapters 3-5

April 6, 2021

The book sections corresponding to each problem are mentioned. The ones covered in Quiz 5 are marked in red. The problems below are meant to illustrate important ideas from the sections above and connect them with each other. ***Many of the questions are more difficult or time consuming or computationally heavy than it would be appropriate for an exam setting but you could still be asked to answer questions similar to parts of them.*** *Some include ideas that aren't absolutely central to the course but do nonetheless appear in the homework and as such they are fair game for exams.*

1. Book sections: 3.5 (parts a-c), 5.7 (parts d-e)

   For which ones of the following non-homogeneous equations/systems can we use the method of undetermined coefficients to find a particular solution? (Primes denote derivatives with respect to x and $D = \frac{\partial}{\partial x}$.) For those for which the method applies find the appropriate form of a particular solution, but do not find the undetermined coefficients.

   (a) $y'' + \sin(x)y' + y = \sin(2x)$.

   (b) $(D-6)(D^2+1)y = x^2\cos(x) + e^x$

   (c) $y'(x) + 5y = \tan(x)$

   (d) $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 15 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \sin(t) \\ t^2e^{8t}+2 \end{bmatrix}$

   (e) $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ -4 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \cos(t) \\ \sinh(t) \end{bmatrix}$

2. Book sections: 3.2 (reduction of order), 3.5 (non-homogeneous equations/variation of parameters)

   You are given the following differential equation (Bessel's equation of order 1/2):

   $$x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0. \tag{1}$$

   You are given the function $y_1 = x^{-\frac{1}{2}}\cos(x)$ on $I = (0, \infty)$.

(a) Check that y_1 is a solution of (1) on I.

(b) Use the method of reduction to find a second linearly independent solution y_2 for (1) on I.

(c) Compute the Wronskian $W(y_1, y_2)$ and confirm that it never vanishes on I.

(d) Use the method of Variation of Parameters to express a particular solution of

$$x^2y'' + xy' + (x^2 - \frac{1}{4})y = \sin(x)$$

as a sum of integrals of known functions (you do not need to actually compute the integrals). *Hint: before you apply the method make sure to review §3.5, Theorem 1 to remind yourself the form of differential equations for which the method of Variation of Parameters applies.* Would it be appropriate to use the method of undetermined coefficients instead?

3. Book sections: 3.2, 3.3 (parts a-c), 4.1 (part d), 5.1 (part g) 5.2 (parts e, f), 5.6 (part h). Parts d and later can be answered without having done parts a-c.

In much of this class we encounter different ways and points of view for dealing with the same type of problem. This question connects some of these ideas.

You are given the following differential equation (prime denotes derivative with respect to t):

$$y^{(3)} - y'' + 4y' - 4y = 0. \tag{2}$$

(a) Find its characteristic equation and compute its roots (search for a small integer root first and find the rest by division of polynomials).

(b) Find a general solution for (2) and use them to solve (2) with the initial conditions

$$y(0) = 1, \quad y'(0) = 4, \quad y''(0) = 0.$$

(c) Your general solution in Part 3b should be a linear combination of 3 particular solutions. Set up their Wronskian (no need to evaluate).

(d) Rewrite (2) in the form of an equivalent 1st order system in matrix form. Your system should be of the form

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \tag{3}$$

for an appropriate 3×3 matrix $\mathbf{A}$.

(e) Compute the characteristic polynomial of $\mathbf{A}$ and its roots. What do you notice?

(f) Write a general solution of (3). *Make sure that the vector valued function in your solution has real entries.*

(g) Your general solution in Part 3f should be a linear combination of 3 particular solutions. Set up their Wronskian (no need to evaluate).

(h) Find a fundamental matrix for the system (3). Using it, compute the matrix $e^{\mathbf{A}t}$ and use it to find the solution of (3) satisfying $\mathbf{x}(0) = [1, 4, 0]^T$ (here T stands for the transpose). It would be helpful to use a Computer Algebra System for some of these computations.

4. Book sections: 3.2, 3.3 (parts a-c), 4.1 (part d), 5.1 (part g) 5.2 (parts e, f), 5.5 (part f), 5.6 (part h)

   Same question as 3, except now replace (2) with

$$y^{(3)} - 5y'' + 8y' - 4y = 0.$$

   Remark: Problems (3) and (4) cover some of the main ingredients included in solving linear constant coefficient ODEs and linear 1st order systems with constant coefficients. If you would like some extra practice on those topics, here are some problems exhibiting a variety of situations regarding the roots of the characteristic equation and the eigenvalues of the system correspondingly:

   (a) Linear Constant Coefficient ODE: §3.3 Problems 4, 12, 15, 29

   (b) 1st order linear systems: §5.2 Problems 12, 13, 18; §5.5 Problems 3, 6, 8

   Those are all similar to problems you have done for your homework.

5. Book section: 4.2

   Use the method of elimination to find the general solution to the following system:

$$\begin{aligned} x' &= 4x + y - 12t \\ y' &= -2x + y \end{aligned} \tag{4}$$

   Remark (not for exams): In the system (4) there is dependence of time on the right hand side, so the phase portrait of the system depends on time (in more geometric language (4) describes the integral curves of a time dependent vector field). You can use Mathematica to enter the following command

```
Animate[VectorPlot[{{4x+y-12t,-2x+y}},{x,-5,5},{y,-5,5}],{t,0,10}]
```

   which produces an animated phase portrait of (4) changing with time.

6. Book section: 5.3

   You are given the following system, where α is a real parameter and primes denote derivatives with respect to time:

$$\begin{aligned} x_1' &= 3x_1 + x_2 \\ x_2' &= \alpha\, x_1 - x_2. \end{aligned} \tag{5}$$

Use the pictures on pages 315-316 in the textbook and eigenvalue computations to decide for what values of α, if any, the phase plane portrait corresponding to the system exhibits

(a) A proper nodal source

(b) An improper nodal source

(c) A spiral sink

(d) A spiral source

(e) A center

(f) A saddle point

(g) Parallel lines

Use some software such as pplane8 to plot the phase plane portrait of (5) for various values of α to check your answers.

7. Book section: 3.3

Find a general solution for the following differential equation given in operator form ($D = \frac{\partial}{\partial x}$):

$$(D-2)(D^2+1)(D^2+4D+8)^2 y = 0.$$

Find a non-trivial (i.e. not identically 0) particular solution y_p with the property $y_p \to 0$ as $x \to \infty$.

8. Book sections: 3.4 (parts a, c) 3.6 (parts b, d, e)

A body with mass 150 g is attached to the end of a horizontal spring that is stretched 15 cm by a force of $9N$.

(a) Suppose that initially the body is held $10cm$ to the left of its equilibrium position and that there is no damping force (i.e. no friction, air resistance etc) or any external force exerted on the body along the horizontal axis besides the force of the spring. Set up an Initial Value Problem (IVP) describing the displacement of the body from equilibrium once it is left free to move with initial velocity $4m/s$ to the right. *Do not forget the initial conditions and make sure that the units of the various quantities involved are consistent.* Then solve your IVP to find a solution of the form $x(t) = C\cos(\omega_0 t - \alpha)$, where $0 \leq \alpha < 2\pi$. What is the period of the motion?

(b) Suppose now that initially the body is at rest at its equilibrium position and that there is no damping force (i.e. no friction, air resistance etc) but we apply a horizontal external force of the form $F(t) = 6\sin(\omega t)$ (in N) for an angular frequency ω (the positive axis is towards the right). Set up an Initial Value Problem (IVP) describing the displacement of the body from equilibrium as a function of time. *Do not forget the initial conditions.* Your Initial Value Problem will depend on the parameter ω. What is the angular frequency ω that would produce a resonance?

(c) Suppose now that there is no external periodic force. At time $t = 0s$ the body is lying $10cm$ to the left of its equilibrium position with velocity 0. We know that there is certain damping force, of the form $F(t) = -cv(t)$, where v is the velocity of the body and $c > 0$ is an unknown damping constant. Set up an IVP describing the displacement of the body from equilibrium as a function of time (your IVP will depend on c). We observe the motion of the body and notice that in the first $1s$ the body passes through its equilibrium position 3 times. What can we say about the motion?

a. It is underdamped
b. It is critically damped
c. It is overdamped
d. There is not enough information given.

Write down a solution to your IVP, depending parametrically on c.

(d) Assume that at time $t = 0s$ the body is lying $10cm$ to the left of its equilibrium position with velocity 0, and that there is a damping force with unknown damping constant $c > 0$ and and external horizontal periodic force of the form $F(t) = 6\cos(\omega t)$. Again, write an IVP for the displacement of the body from equilibrium. Find the amplitude of the steady periodic solution as a function of ω when $c = 2$ and when $c = 5$. Graph the amplitude of the steady periodic solution as a function of ω in those two cases. In which of the two cases do we have practical resonance and at what angular frequency ω does that happen?

(e) * Your friend is arguing that when $c = 2$ the solution of the IVP in part d) is the sum of the (transient) solution of the IVP in part c) and a steady periodic function of t of the form $C\cos(\omega t - \alpha)$. Are they correct?