1. <u>Ans:</u> b), d), e)

We need const coef equation/system and linear comb. of products of sin/cos, exponential, polynomial

<u>Particular Solutions:</u>

For b): First try for part. sol'n:

$$y_1 = (a_1 x^2 + b_1 x + c_1)\cos(x) + (a_2 x^2 + b_2 x + c_2)\sin(x) + d e^x$$

Find complementary sol'n to check for duplication:

Char. eq'n:

$$(r-6)(r^2+1)=0 \Rightarrow \begin{cases} r = 6 \\ r = \pm i \end{cases}$$

General sol'n: $c_1 e^{6t} + c_2 \cos(t) + c_3 \sin(t)$

So we have duplication.

So finally take:

$$y_p = x y_1$$

! It is also valid to only multiply by x the problematic terms, i.e. set

$$y_p = x(a_1x^2+b_1x+c_1)\cos(x) + x(a_2x^2+b_2x+c_2)\sin(x)$$
$$+ d e^x$$
no x here!

d) Non-homogeneous term can be written as
$$\sin(t)\begin{bmatrix}1\\0\end{bmatrix} + t^2e^{8t}\begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}0\\2\end{bmatrix}$$
First guess
$$y_1 = \begin{bmatrix}a_1\\a_2\end{bmatrix}\sin(t) + \begin{bmatrix}b_1\\b_2\end{bmatrix}\cos(t) + \begin{bmatrix}c_1\\c_2\end{bmatrix}e^{8t} + \begin{bmatrix}d_1\\d_2\end{bmatrix}te^{8t}$$
$$+ \begin{bmatrix}e_1\\e_2\end{bmatrix}t^2e^{8t} + \begin{bmatrix}f_1\\f_2\end{bmatrix} \qquad 1$$
Compl. Sol'n (checking for duplication)
$$\det\begin{pmatrix}2-\lambda & 15\\ 4 & -2-\lambda\end{pmatrix} = \lambda^2 - 64 \Rightarrow \lambda = \pm 8$$
So compl. sol'n of the form
$$\underline{v}_1 e^{8t} + \underline{v}_2 e^{-8t}$$
where $\underline{v}_1, \underline{v}_2$ eigenvectors corr. to $8, -8$

So there is duplication. Take

$$y_P = \begin{bmatrix}a_1\\a_2\end{bmatrix}\sin(t) + \begin{bmatrix}b_1\\b_2\end{bmatrix}\cos(t) + \begin{bmatrix}c_1\\c_2\end{bmatrix}te^{8t} + \begin{bmatrix}d_1\\d_2\end{bmatrix}t^2e^{8}$$
$$+ \begin{bmatrix}e_1\\e_2\end{bmatrix}t^3e^{8t} + \begin{bmatrix}g_1\\g_2\end{bmatrix}e^{8t} + \begin{bmatrix}f_1\\f_2\end{bmatrix}$$

Explanation: Multiplied the terms underlined with blue in 1 by $t$ to avoid duplication, but then also included the lower order term underlined with red. As before, we only multiplied the problematic terms (i.e. those whose derivatives appear in the compl. sol'n) by $t$.

e) Initial guess:

$$y_1 = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cos(t) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \sin(t) + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \sinh(t)$$
$$+ \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \cosh(t)$$

Complem. sol'n:

$$\det \begin{bmatrix} 5-\lambda & 7 \\ -4 & -3 \end{bmatrix} = t^2 - 2t + 13$$
$$= 1 \pm 2\sqrt{3} i$$

So no duplication and $y_p = y_1$ works; the complementary sol'n is of the form

$$\underline{\underline{v}}_1 e^t \cos(2\sqrt{3} t) + \underline{\underline{v}}_2 e^t \sin(2\sqrt{3} t)$$

for appropriate $\underline{v}_1, \underline{v}_2$.

(Recall: to find $\underline{v}_1, v_2$ take real & imaginary pt of $e^{(1+2\sqrt{3})t}\underline{v}$, where $\underline{v}$ eigenv. corresponding to $1+2\sqrt{3}i$ w/ complex entries)

2. a) $$y_1'' = \frac{3}{4}x^{-\frac{5}{2}}\cos(x) + x^{-\frac{3}{2}}\sin(x) - x^{-\frac{1}{2}}\cos(x)$$

$$y_1' = -\frac{1}{2}x^{-\frac{3}{2}}\cos(x) - x^{-\frac{1}{2}}\sin(x)$$

So $x^2y'' + xy' + (x^2 - \frac{1}{4})y$

$$= \frac{3}{4}\cancel{x^{-\frac{1}{2}}\cos(x)} + \cancel{x^{\frac{1}{2}}\sin(x)} - \cancel{x^{\frac{3}{2}}\cos(x)}$$

$$-\frac{1}{2}\cancel{x^{-\frac{1}{2}}\cos(x)} - \cancel{x^{\frac{1}{2}}\sin(x)}$$

$$+\cancel{x^{\frac{3}{2}}\cos(x)} - \frac{1}{4}\cancel{x^{-\frac{1}{2}}\cos(x)} = 0$$

b) set $y_2 = u y_1$

$$x^2\left(u''y_1 + 2u'y_1' + uy_1''\right)$$

$$+ x\left(u'y_1 + uy_1'\right) + \left(x^2 - \frac{1}{4}\right) = 0$$

$$\Rightarrow u\overbrace{\left(x^2 y_1'' + xy_1' + \left(x^2 - \frac{1}{4}\right)\right)}^{0}$$

$$+ 2x^2u'y_1' + x^2u''y_1 + xu'y_1 = 0$$

$$\Rightarrow u''\,x^2y_1 + u'\left(2x^2y_1' + xy_1\right) = 0$$

$$\Rightarrow \frac{u''}{u'} = -\frac{2x^2y_1' + xy_1}{x^2y_1}$$

Plug in $y_1 = x^{-\frac{1}{2}}\cos(x)$

$$\frac{u''}{u'} = -\frac{2x^2\left(-\frac{1}{2}x^{-\frac{3}{2}}\cos(x) - x^{-\frac{1}{2}}\sin(x)\right) + x^{\frac{1}{2}}\cos(x)}{x^{3/2}\cos(x)}$$

$$= \frac{2x^{3/2}\sin(x)}{x^{3/2}\cos(x)}$$

$$= \frac{2\sin(x)}{\cos(x)}$$

$$\Rightarrow \quad \ln(u') = -2\ln(\cos(x)) + C_1$$

$$\Rightarrow \quad u' = \frac{C_1}{\cos^2(x)} \quad \Rightarrow \quad u = C_1\tan(x) + C_2$$

Take $u = \tan(x)$ for simplicity,

so $y_2 = x^{-\frac{1}{2}}\sin(x)$.

c) $y_2' = -\frac{1}{2}x^{-\frac{3}{2}}\sin(x) + x^{-\frac{1}{2}}\cos(x)$

So

$$W(y_1,y_2) = \begin{vmatrix} x^{-\frac{1}{2}}\cos(x) & x^{-\frac{1}{2}}\sin(x) \\ -\frac{1}{2}x^{-\frac{3}{2}}\cos(x) - x^{-\frac{1}{2}}\sin(x) & -\frac{1}{2}x^{-\frac{3}{2}}\sin(x) + x^{-\frac{1}{2}}\cos(x) \end{vmatrix}$$

$$= -\frac{1}{2}x^{-2}\sin(x)\cos(x) + x^{-1}\cos^2(x)$$
$$+ \frac{1}{2}x^{-2}\sin(x)\cos(x) + x^{-1}\sin^2(x)$$
$$= x^{-1} \neq 0 \text{ on } I$$

d) To use Variation of Parameters

Rewrite

$$y'' + x^{-1}y' + \left(1 - \frac{1}{4x^2}\right)y = \frac{\sin(x)}{x^2}$$

Want the coefficient to be 1.

$$y_p(x) = -x^{-\frac{1}{2}}\cos(x)\int \frac{x^{-\frac{1}{2}}\sin(x)}{x^{-1}}\frac{\sin(x)}{x^2}\,dx +$$

$$x^{-\frac{1}{2}}\sin(x)\int \frac{x^{-\frac{1}{2}}\cos(x)\sin(x)}{x^{-1}\quad x^2}\,dx$$

$$= -x^{-\frac{1}{2}}\cos(x)\int x^{-\frac{3}{2}}\sin^2(x)\,dx + x^{-\frac{1}{2}}\sin(x)\int x^{-\frac{3}{2}}\cos(x)\sin(x)\,dx$$

The method of undetermined coefficients is not appropriate bec. the given eq'n is not a const. coef. eq'n.

3. $y^{(3)} - y'' + 4y' - 4y = 0$

a)

Char. Eq'n: $r^3 - r^2 + 4r - 4 = 0$

$r = 1$ is a root

w/ long division:

$$r^3 - r^2 + 4r - 4 = (r-1)(r^2+4)$$

So roots: $r = 1,\quad r = \pm 2i$

b) Gen. sol'n

$$y = c_1 e^t + c_2\cos(2t) + c_3\sin(2t)$$

So:

$$y'(t) = c_1 e^t - 2c_2\sin(2t) + 2c_3\cos(2t)$$

$$y''(t) = c_1 e^t - 4c_2\cos(2t) - 4c_3\sin(2t)$$

Now at $t=0$ want $y(0)=1$, $y'(0)=4$, $y''(0)=0$

$$\Rightarrow \begin{cases} c_1 + c_2 = 1 & (1) \\ c_1 + 2c_3 = 4 & (2) \\ c_1 - 4c_2 = 0 & (3) \end{cases}$$

(3), (1) $\Rightarrow$ $c_2 = \frac{1}{5}$, $c_1 = \frac{4}{5}$

(2) $\Rightarrow$ $c_3 = \frac{16}{10}$

So

$$y(t) = \frac{4}{5}e^t + \frac{1}{5}\cos(2t) + \frac{16}{10}\sin(2t).$$

c)

$$W(e^t, \cos(2t), \sin(2t)) = \begin{vmatrix} e^t & \cos(2t) & \sin(2t) \\ e^t & -2\sin(2t) & 2\cos(2t) \\ e^t & -2\cos(2t) & -4\sin(2t) \end{vmatrix}$$

d) set $x_1 = y$, $x_2 = y'$, $x_3 = y''$

$$x_1' = x_2$$
$$x_2' = x_3$$
$$x_3' = y'' - 4y' + 4y = x_3 - 4x_2 + 4x_1$$

So

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -4 & 1 \end{bmatrix}}_{= \underline{\underline{A}}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

e)

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 4 & -4 & 1-\lambda \end{vmatrix} \leftarrow \text{expand here}$$

$$= 4 \cdot 1 + 4(-\lambda) + (1-\lambda)\lambda^2$$
$$= -\lambda^3 + \lambda^2 - 4\lambda + \lambda.$$

Setting $=0$ we find the same eq'n as the characteristic eq'n in part a).
So eigenvalues: $\lambda = 1$, $\lambda = \pm 2i$

<u>Find eigenvectors:</u> $\lambda = 1$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 4 & -4 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} v_2 = v_3 \\ v_1 = v_2 \end{cases}$$

So take $\underline{\underline{v}}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, find one sol'n $\underline{\underline{x}}_1 = e^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

For $\lambda = 2i$

$$\begin{array}{c} (1) \\ (2) \\ (3) \end{array} \begin{bmatrix} -2i & 1 & 0 \\ 0 & -2i & 1 \\ 4 & -9 & 1-2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{r} (1)\cdot\frac{1}{-2i} \to (1) \\ (2) \\ (3)\cdot\frac{1}{4} \to (3) \end{array} \begin{bmatrix} 1 & i/2 & 0 \\ 0 & -2i & 1 \\ 1 & -1 & \frac{1}{4}-\frac{i}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{r} (1) \\ (2) \\ (3)-(1) \to (3) \end{array} \begin{bmatrix} 1 & i/2 & 0 \\ 0 & -2i & 1 \\ 0 & -1-\frac{i}{2} & \frac{1}{4}-\frac{i}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{r} (1) \\ (2)\cdot\frac{1}{-2i} \to (2) \\ (3)\cdot\frac{2}{-2-i} \to (3) \end{array} \begin{bmatrix} 1 & i/2 & 0 \\ 0 & 1 & i/2 \\ 0 & 1 & i/2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so $v_2 = -\frac{i}{2} v_3$

$v_1 = -\frac{i}{2} v_2 \Rightarrow v_1 = -\frac{1}{4} v_3$

Eigenvector:

$$\underline{V}_2 = \begin{bmatrix} -\frac{1}{4} \\ -\frac{i}{2} \\ 1 \end{bmatrix} \qquad \text{(multiply by 4)}$$

Sol'n:

$$\underline{x}_2 = e^{2it} \begin{bmatrix} -1 \\ -2i \\ 4 \end{bmatrix}$$

Take real & imaginary pts to find sols w/

real entries:

$$\underline{\underline{x}}_2 = \left(\cos(2t) + i\sin(2t)\right)\begin{bmatrix}-1\\-2i\\4\end{bmatrix}$$

$$= \begin{bmatrix}-\cos(2t) & -i\sin(2t)\\ 2\sin(2t) & -2i\cos(2t)\\ 4\cos(2t) & +4i\sin(2t)\end{bmatrix}$$

$$= \begin{bmatrix}-\cos(2t)\\2\sin(2t)\\4\cos(2t)\end{bmatrix} + i\begin{bmatrix}-\sin(2t)\\-2\cos(2t)\\4\sin(2t)\end{bmatrix}$$

So general sol'n

$$\underline{\underline{x}} = c_1 e^t\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}-\cos(2t)\\2\sin(2t)\\4\cos(2t)\end{bmatrix} + c_3\begin{bmatrix}-\sin(2t)\\-2\cos(2t)\\4\sin(2t)\end{bmatrix}$$

g) $$W = \begin{vmatrix}e^t & -\cos(2t) & -\sin(2t)\\ e^t & 2\sin(2t) & -2\cos(2t)\\ e^t & 4\cos(2t) & 4\sin(2t)\end{vmatrix}$$

Notice that the Wronskian agrees w/ part c.

h) Fund. matrix:

$$\Phi(t) = \begin{bmatrix}e^t & -\cos(2t) & -\sin(2t)\\ e^t & 2\sin(2t) & -2\cos(2t)\\ e^t & 4\cos(2t) & 4\sin(2t)\end{bmatrix}$$

$$e^{At} = \Phi(t)\,\Phi(0)^{-1}$$

$$= \begin{bmatrix} e^t & -\cos(2t) & -\sin(2t) \\ e^t & 2\sin(2t) & -2\cos(2t) \\ e^t & 4\cos(2t) & 4\sin(2t) \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -2 \\ 1 & 4 & 0 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \frac{4}{5}e^t + \frac{1}{5}\cos(2t) - \frac{2}{5}\cos(2t) & \frac{1}{2}\sin(2t) & \frac{e^t}{5} - \frac{1}{5}\cos(2t) - \frac{1}{10}\sin(2t) \\ \frac{4}{5}e^t - \frac{4}{5}\cos(2t) - \frac{2}{5}\sin(2t) & \cos(2t) & \frac{e^t}{5} - \frac{1}{5}\cos(2t) + \frac{2}{5}\sin(2t) \\ \frac{4}{5}e^t - \frac{4}{5}\cos(2t) + \frac{8}{5}\sin(2t) & -2\sin(2t) & \frac{e^t}{5} + \frac{4}{5}\cos(2t) + \frac{2}{5}\sin(2t) \end{bmatrix}$$

(using Mathematica)

Now the required sol'n is $\underline{x}(t) = e^{At}\begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} \frac{4}{5}e^t + \frac{1}{5}\cos(2t) + \frac{8}{5}\sin(2t) \\ \frac{4}{5}e^t + \frac{16}{5}\cos(2t) - \frac{2}{5}\sin(2t) \\ \frac{4}{5}e^t - \frac{4}{5}\cos(2t) - \frac{32}{5}\sin(2t) \end{bmatrix} \leftarrow$$

the sol'n in part b), as it should be!

4. $y^{(3)} - 5y'' + 8y' - 4y = 0$

a) $r^3 - 5r^2 + 8r - 4 = 0$

Coef. add up to $0$ so $r = 1$ is a sol'n.

$$r^3 - r^2 - 4r^2 + 4r + 4r - 4 = 0$$

$$\Rightarrow r^2(r-1) - 4r(r-1) + 4(r-1) = 0$$

$$\Rightarrow (r^2 - 4r + 4)(r-1) = 0$$

$$\Rightarrow (r-2)^2(r-1) = 0$$

$$\Rightarrow r = 1,\ 2 \ \text{(repeated)}$$

b) General sol'n:

$$y(t) = c_1 e^t + c_2 e^{2t} + c_3 te^{2t}$$
$$y'(t) = c_1 e^t + 2c_2 e^{2t} + 2c_3 t e^{2t} + c_3 e^{2t}$$
$$y''(t) = c_1 e^t + 4c_2 e^{2t} + 4c_3 t e^{2t} + 4c_3 e^{2t}$$

$$y(0) = 1 \Rightarrow \quad c_1 + c_2 = 1 \qquad (1)$$
$$y'(0) = 4 \Rightarrow \quad c_1 + 2c_2 + c_3 = 4 \qquad (2)$$
$$y''(0) = 0 \Rightarrow \quad c_1 + 4c_2 + 4c_3 = 0 \qquad (3)$$

$$(1),(2) \Rightarrow \quad c_2 + c_3 = 3$$
$$(1),(3) \Rightarrow \quad 3c_2 + 4c_3 = -1$$
$$9 - 3c_3 + 4c_3 = -1 \Rightarrow c_3 = -10$$

so $c_2 = 13$, $c_1 = -12$

$$y(t) = -12e^t + 13e^{2t} - 10te^{2t} \quad (*)$$

c)
$$W(e^t, e^{2t}, te^{2t}) = \begin{vmatrix} e^t & e^{2t} & te^{2t} \\ e^t & 2e^{2t} & 2te^{2t} + e^{2t} \\ e^t & 4e^{2t} & 4te^{2t} + 4e^{2t} \end{vmatrix}$$

d) Set
$$\begin{cases} x_1 = y \\ x_2 = x_1' = y' \\ x_3 = y'' = x_2' \end{cases}$$

then

$$x_1' = x_2$$
$$x_2' = x_3$$
$$x_3' = 4x_1 - 8x_2 + 5x_3$$

$$\underline{\underline{x}}' = \underline{\underline{A}}\,\underline{\underline{x}}, \quad \underline{\underline{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \underline{\underline{A}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -8 & 5 \end{bmatrix}$$

e) Set

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 4 & -8 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda(-5\lambda + \lambda^2 + 8) + 4 = 0$$

$$\Rightarrow -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$$

This is the same as the characteristic eq'n of the dif. eq'n given, so $\lambda = 1, 2$ (repeated).

f) $$(A - I)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \underline{0} \Rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 4 & -8 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \underline{0}$$

4x first add to third

$$\Rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -4 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} v_1 = v_2 \\ v_2 = v_3 \end{cases}$$

$\Rightarrow$ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector and $e^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ a sol'n.

$$(A - 2I) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \underline{0} \Rightarrow \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & -4 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \underline{0}$$

$$\Rightarrow \begin{cases} v_2 = 2v_1 \\ v_3 = 2v_2 = 4v_1 \end{cases} \quad \text{so} \quad \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \text{ works}$$

So can't find 2 lin indep. eigenvectors.

Look for generalized ones

$$(A - 2I)^2 = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & -4 & 2 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & -4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -4 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So:

$$(A - 2I)^2 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \Rightarrow v_3 = -4v_1 + 4v_2$$

Want also

$$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & -4 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \neq \underline{\underline{0}}$$

take $\quad v_2 = 1, \ v_1 = 1, \ v_3 = 0$

Then

$$\underbrace{\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & -4 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{} = \begin{bmatrix} -1 \\ -2 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \underline{\underline{v}}_2, \qquad \begin{bmatrix} -1 \\ -2 \\ -4 \end{bmatrix} = \underline{\underline{v}}_1$$

So lin. indep. sols:

$$\underline{\underline{x}}_1 = e^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \underline{\underline{x}}_2 = e^{2t}\begin{bmatrix} -1 \\ -2 \\ -4 \end{bmatrix}, \quad \underline{\underline{x}}_3 = \left(\begin{bmatrix} -1 \\ -2 \\ -4 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) e^{2t}$$

General sol'n

$$\underline{\underline{x}} = c_1 \underline{\underline{x}}_1 + c_2 \underline{\underline{x}}_2 + c_3 \underline{\underline{x}}_3$$

g)

$$W(\underline{\underline{x}}_1, \underline{\underline{x}}_2, \underline{\underline{x}}_3) = \begin{vmatrix} e^t & -e^{2t} & (1-t)e^{2t} \\ e^t & -2e^{2t} & (1-2t)e^{2t} \\ e^t & -4e^{2t} & -4e^{2t} \end{vmatrix}$$

h) $\underline{\underline{e^{At}}} = \underline{\underline{\Phi}}(t)\,\underline{\underline{\Phi}}(0)^{-1}$, where $\underline{\underline{\Phi}}$ is a fundamental matrix.

Take
$$\Phi(t) = \begin{bmatrix} e^t & -e^{2t} & (1-t)e^{2t} \\ e^t & -2e^{2t} & (1-2t)e^{2t} \\ e^t & -4e^{2t} & (-4t)e^{2t} \end{bmatrix}$$

$$\Phi(0) = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 1 \\ 1 & -4 & 0 \end{bmatrix}$$

$$\Rightarrow \Phi(0)^{-1} = \begin{bmatrix} 4 & -4 & 1 \\ 1 & -1 & 0 \\ -2 & 3 & -1 \end{bmatrix} \quad \text{(using a CAS)}$$

So
$$e^{At} = \begin{bmatrix} 4e^t - e^{2t} - 2e^{2t}(1-t) & -4e^t + e^{2t} + 3e^{2t}(1-t) & e^t - e^{2t}(1-t) \\ 4e^t - 2e^{2t} - 2e^{2t}(1-2t) & -4e^t + 2e^{2t} + 3e^{2t}(1-2t) & e^t - e^{2t}(1-2t) \\ 4e^t - 4e^{2t} + 8e^{2t}t & -4e^t + 4e^{2t} - 12e^{2t} & e^t + 4e^{2t}t \end{bmatrix}$$

and the solution we seek is

$$e^{At}\begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} -e^t(12 + e^t(-13 + 10t)) \\ -4e^t(3 + e^t(-4 + 5t)) \\ -4e^t(3 + e^t(-3 + 10t)) \end{bmatrix}$$

The 1st row is

$$x_1(t) = -12e^{t} + 13e^{2t} - 10te^{2t}$$

which agrees w/ ⊛, as it should.

5. $x' = 4x + y - 12t$
$y' = -2x + y$ $\boxed{2}$

Write in pol. dif. op'tor notation:

$$(D-4)x - y = -12t \qquad L_1 = D-4,\ L_2 = -1$$
$$2x + (D-1)y = 0 \qquad L_3 = 2,\ L_4 = D-1$$

Apply method of elimination:

$$2(D-4) - 2y = -24t$$
$$(D-4)\cdot 2x + (D-4)(D-1)y = 0 \quad \ominus$$

$$-2y - (D-4)(D-1)y = -24t$$
$$(D^2 - 5D + 4)y + 2y = 24t$$
$$\Rightarrow \underbrace{(D^2 - 5D + 6)y = 24t}_{} \quad ⊛$$

non-homog. const. coef., can use Undet coef.

Compl. sol'n:

$$(D^2 - 5D + 6)y = 0 \overset{\text{char.}}{\underset{\text{eq'n}}{\Rightarrow}} r^2 - 5r + 6 = 0$$
$$\Rightarrow r = 2,\ r = 3$$

$$y_c = C_1 e^{3t} + C_2 e^{2t}$$

Guess for part. sol'n:

$$y = At + B$$

Plug into ⊛

$$-5A + 6(At+B) = 24t$$

$$\Rightarrow \begin{cases} -5A + 6B = 0 \\ 6A = 24 \end{cases}$$

$$\Rightarrow A = 4, \quad B = \frac{10}{3}$$

So:

$$y = c_1 e^{3t} + c_2 e^{2t} + 4t + \frac{10}{3}$$

To find x:

$\boxed{2}$ $\Rightarrow \quad x = \frac{y - y'}{2}$

$$\Rightarrow x = \frac{c_1 e^{3t} + c_2 e^{2t} + 4t + \frac{10}{3} - 3c_1 e^{3t} - 2c_2 e^{2t} - 4}{2}.$$

6. We compute eigenvalues:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \underbrace{\begin{bmatrix} 3 & 1 \\ \alpha & -1 \end{bmatrix}}_{\underline{\underline{A}}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \begin{vmatrix} 3-\lambda & 1 \\ \alpha & -1-\lambda \end{vmatrix}$$

$$= -3 - 3\lambda + \lambda + \lambda^2 - \alpha$$

$$= \lambda^2 - 2\lambda - 3 - \alpha$$

So:

$$\lambda = \frac{2 \pm \sqrt{4 + 4(\alpha+3)}}{2}$$

$$= 1 \pm \sqrt{1 + \alpha + 3}$$

$$= 1 \pm \sqrt{4+\alpha}$$

If $\underline{\alpha > -4}$ $\rightarrow$ 2 real roots, distinct.

One is always $>0$.

$\rightarrow$ If $\sqrt{4+\alpha} > 1 \Rightarrow \alpha > -3$

there are 2 roots of opposite sign.

$\rightarrow$ $\sqrt{4+\alpha} = 1$ we have one 0 one positive e.v

$\rightarrow$ $\sqrt{4+\alpha} < 1 \Leftrightarrow -4 < \alpha < -3$

two positive distinct roots.

If $\underline{\alpha = -4}$ $\rightarrow$ repeated root $\lambda = 1$

Find defect:

$$(\underline{A} - I)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow v_2 = -2v_1$$

So defect 1 eigenvalue

If $\alpha < -4$     2 distinct conjugate cplx roots w/ positive real part.

Look @ figure 5.3.16 in book.

a) For Proper nodal source we want repeated positive eigenvalue of defect 0, so no $\alpha$ gives that

b) Improper nodal source → either 2 distinct positive roots or positive repeated of defect 1.

So:

$$-4 \leq \alpha < -3$$

c) Spiral sink needs cplx conj. w/ neg. real pt so no $\alpha$ works

d) Spiral source → cplx conj w/ pos. real pt so $\alpha < -4$ works.

e) Center needs purely imaginary roots so no $\alpha$

f) Saddle need 2 distinct real e.v. of opposite sign so $\alpha > -3$

g) Parallel lines occur when either one 0 & one pos/neg. e.v. or defective e.v. 0. so $\alpha = -3$

7. Char. eq'n:

$$(r-2)(r^2+1)(r^2+2r+8)^2 = 0$$

$r = 2$, $r = \pm i$

$$r = \frac{-2 \pm \sqrt{4-32}}{2}$$

$= -1 \pm i\sqrt{7}$, each repeated twice.

So

$$y(x) = c_1 e^{2x} + c_2 e^{-x}\cos(\sqrt{7}x) + c_3 e^{-x}\sin(\sqrt{7}x)$$
$$+ c_4 x e^{-x}\cos(\sqrt{7}x) + c_5 x e^{-x}\sin(\sqrt{7}x)$$
$$+ c_6 \cos(x) + c_7 \sin(x)$$

To ensure that $y(x) \to 0$ as $x \to \infty$ it suffices to take $c_1 = c_6 = c_7 = 0$.
Any choice of the other constants works
e.g take $y(x) = x e^{-x}\cos(\sqrt{7}x)$

8. Stretch 15 cm by force of 9N

$\Rightarrow \quad 9 = K \cdot 0.15 \Rightarrow \quad k = 60\ N/m$

a) IVP : $m = 0.15$ kg, $k = 60 N/m$

(Recall : $1N = 1\ kgm/s^2$)

so

$$mx'' + kx = 0 \Rightarrow \begin{cases} 0.15x'' + 60x = 0 \\ x(0) = -0.1 \\ x'(0) = 4 \end{cases}$$

Units for $x$ are in m.

$$x'' + 400x = 0 \Rightarrow r^2 + 400 = 0$$
$$\Rightarrow r = \pm i20$$

so

$$x(t) = A\cos(20t) + B\sin(20t)$$

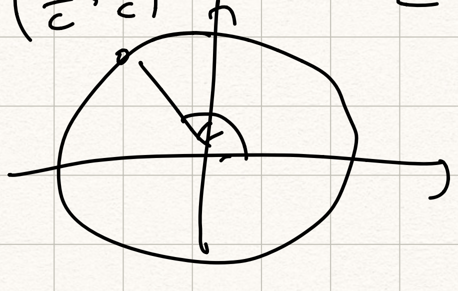
$x(0) = -0.1 \rightarrow \quad A = -0.1$

$x'(0) = 4 \rightarrow \quad 20B = 4 \Rightarrow B = 0.2$

so $\quad \sqrt{A^2 + B^2} = 0.1 \cdot \sqrt{5}$

$$\frac{A}{C} = -\frac{1}{\sqrt{5}}, \quad \frac{B}{C} = \frac{2}{\sqrt{5}}$$

$\left(\frac{A}{C}, \frac{B}{C}\right)$




Second quadrant so phase shift

$$\alpha = \arctan\left(\frac{B}{A}\right) + \pi = \arctan(-2) + \pi$$

So

$$x(t) = 0.1\sqrt{5}\cos\left(20t - \pi - \arctan(-2)\right)$$

Period: $T = \frac{2\pi}{\omega} = \frac{\pi}{10}$ s

b) IVP:

$$\begin{cases} 0.15x'' + 60x = 6\sin(\omega t) \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

We have resonance when $\omega = \omega_0$, where $\omega_0$ was found to be $20$ rad/s in part a). It is the natural frequency of the system.

c)

$$\begin{cases} 0.15x'' + cx' + 60x = 0 \\ x(0) = -0.1 \\ x'(0) = 0 \end{cases}$$

If it passes through equilibrium 3 times then it must be critically damped, otherwise it would pass through equil. position at most once.

Solve IVP

$$0.15r^2 + cr + 60 = 0$$

$$r = \frac{-c \pm \sqrt{c^2 - 9}}{0.3}$$

Underdamped so $c^2 < 9 \Rightarrow$

$$r = -\frac{c}{0.3} \pm i\,\frac{\sqrt{9-c^2}}{0.3}$$

So:

$$x(t) = e^{-\frac{c}{0.3}t}\left(A\cos\left(\frac{\sqrt{9-c^2}}{0.3}t\right) + B\sin\left(\frac{\sqrt{9-c^2}}{0.3}t\right)\right)$$

$$x(0) = A \Rightarrow A = -0.1$$

$$x'(0) = -\frac{c}{0.3}A + \frac{\sqrt{9-c^2}}{0.3}B$$

$$\Rightarrow B = -\frac{c}{3}\,\frac{0.3}{\sqrt{9-c^2}}$$

$$= \frac{-0.1c}{\sqrt{9-c^2}}$$

So

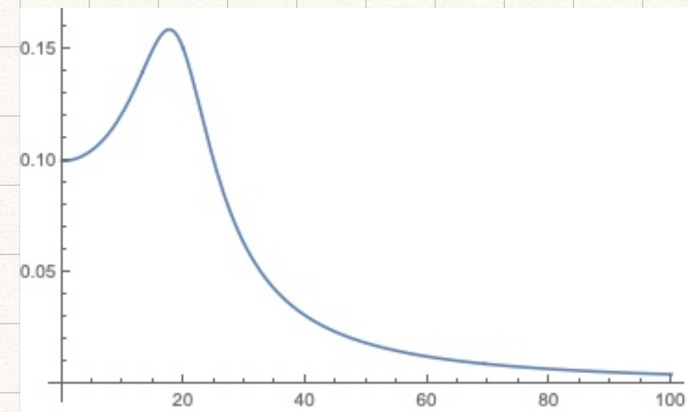
$$x(t) = e^{-\frac{c}{0.3}t}\left(-0.1\cos\left(\frac{\sqrt{9-c^2}}{0.3}t\right) - 0.1\frac{c}{\sqrt{9-c^2}}\sin\left(\frac{\sqrt{9-c^2}}{0.3}t\right)\right)$$

d) $$\begin{cases} 0.15x'' + cx' + 60 = 6\cos(\omega t). \\ x(0) = -0.1 \\ x'(0) = 0 \end{cases}$$

From book amplitude is given by

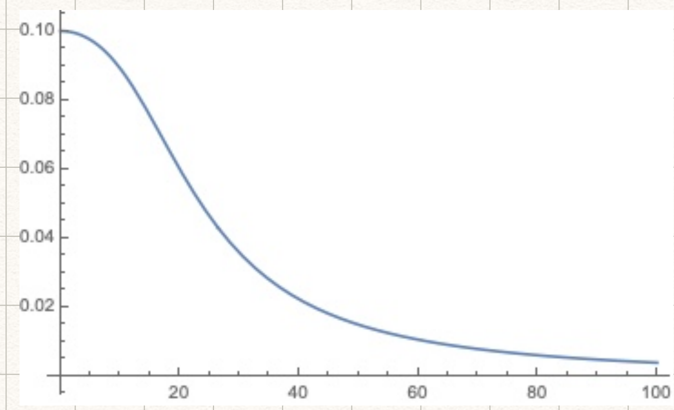
$$C(\omega) = \frac{6}{\sqrt{(60 - 0.15\cdot\omega^2)^2 + (c\omega)^2}}$$

when $c = 2$:




practical resonance

When $c = 5$




No practical resonance

To find the frequency that gives practical resonance when $c = 2$ take derivative of $C(\omega)$ and see when it vanishes.

$$C'(\omega) = 6 \cdot \left(-\frac{1}{2}\right) \overbrace{\left(\sqrt{(60 - 0.15 \cdot \omega^2)^2 + 4\omega^2}\right)^{-\frac{3}{2}}}^{<0}$$

$$\times \left(2(60 - 0.15\omega^2) \cdot (-0.3\omega) + 8\omega\right)$$

$$2(60-0.15\omega^2)(-0.3\omega)+8\omega=0$$
$$\Rightarrow \quad -0.6(60-0.15\omega^2)+8=0$$
$$\Rightarrow \quad \omega \sim 17.6$$

e) The IVP's in parts c) & d) have the same initial conditions. If the statement was true then $y(t) = C\cos(\omega t-\alpha)$ would have to satisfy $y(0) = y'(0) = 0$ and this can't happen unless $C=0$.

What is true is that the sol'n to p. d for $c=2$ is a sum of a periodic function
$$y(t) = C\cos(\omega t-\alpha)$$
and a sol'n of the differential equation in p. c), but w/ different initial conditions.