MA 30300
The Dirichlet problem in polar domains

We learn how to use separation of variables to solve the Dirichlet problem on a domains R in the plane which can be expressed easily in polar coordinates (r, θ) with $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Let R be of the form

$$R = \{(r, \theta) : \alpha < r < \beta, 0 \le \theta < 2\pi\}.$$

Here we will work with the case where $\alpha > 0$, $\beta < \infty$ (annulus). In the textbook you can find a solved example with $\alpha = 0$ (disk[1]), and in the homework you will solve the Dirichlet problem in the exterior of a disk, so with $\alpha > 0$ and $\beta = \infty$. In polar coordinates (r, θ), the Laplacian takes the form

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

Start with the following warm-up problems:

Problem 1 (Warm-up). *Let λ be a real number. We would like to find a non-trivial function $\Theta(\theta)$ satisfying the following:*

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0 \text{ for } \theta \in \mathbb{R} \tag{1}$$

$$\Theta(\theta) = \Theta(\theta + 2\pi). \tag{2}$$

For what values of λ is this possible? What are the corresponding solutions?
Hint: consider separately the cases $\lambda < 0$, $\lambda = 0$, $\lambda > 0$ and find the corresponding general solutions of (1) *and determine for what λ eq.* (2) *can be true.*

Problem 2 (Warm-up). *Find the general solution $R(r)$ to the following problem:*

$$r^2 R'' + rR' = 0. \tag{3}$$

Hint: set $u = R'$ and notice that the resulting differential equation is separable.

Problem 3 (Warm-up). *Now look at the following problem:*

$$r^2 R'' + rR' - n^2 R = 0, \tag{4}$$

where n is a positive integer. This is a second order linear equation with non-constant coefficients; we have not seen a general method for solving such an equation. If you can find two linearly independent solutions R_1 and R_2, any other solution can be written as $R = AR_1 + BR_2$.

Find two linearly independent solutions in the following way: make the educated guess that the solutions will be of the form $R = r^k$, plug into the equation and find the k for which the equation is satisfied.

Let $R = \{(r, \theta) : \alpha < r < \beta, 0 \le \theta < 2\pi\}$ be an annulus as before, $\alpha > 0$. Now we would like to find $u(r, \theta)$ which is 2π periodic in θ and solves the problem

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad \text{on } R \tag{5}$$

$$u(\alpha, \theta) = f(\theta), \quad 0 \le \theta < 2\pi \tag{6}$$

$$u(\beta, \theta) = 0, \quad 0 \le \theta < 2\pi \tag{7}$$

[1]Technically the domain becomes a punctured disk, but by assuming that your solution is continuous at the origin you are actually solving the Dirichlet problem in a disk.

We seek a solution in the form

$$u(r,\theta) = \sum_{n=0}^{\infty} c_n u_n(r,\theta)$$

as usual. We would like to ensure that u_n satisfy (5) and (7). We will determine c_n at the end so that the sum satisfies (6) as well.

Follow the steps below to find such a solution:

1. Assume that $u_n(r,\theta) = R_n(r)\Theta_n(\theta)$. Plug u_n into the PDE (5) and separate variables (all functions of r on the left, all functions of θ on the right).

2. Deduce that Θ_n satisfies an ODE of the form (1). Since we want $u(r,\theta) = u(r,\theta + 2\pi)$, we would like the same to be true for u_n. Conclude that Θ_n also satisfies (2).

3. Use the result of Problem 1 to deduce that $\lambda = 0$ or $\lambda = n^2$, for n a positive integer.

4. Deduce that R_n must satisfy either (3) or (4). Determine the form of R_n so that u_n also satisfies (7).

5. Use the conclusions of Problems 1-3 to deduce that

$$u(r,\theta) = A_0 \ln\left(\frac{r}{\beta}\right) + \sum_{n=1}^{\infty} \left((r/\beta)^n - (r/\beta)^{-n}\right)\left(A_n \cos(n\theta) + B_n \sin(n\theta)\right) \tag{8}$$

6. Now plug in $r = \alpha$ into (8). You want (6) to be true. Recall that $f(\theta)$ is a periodic function of period 2π. What should the coefficients A_0, A_n and B_n be?
   Hint: recall that f has a Fourier series expansion.