MA 30300
The Dirichlet problem in polar domains

We learn how to use separation of variables to solve the Dirichlet problem on a domains $R$ in the plane which can be expressed easily in polar coordinates $(r, \theta)$ with $x = r \cos(\theta)$ and $y = r \sin(\theta)$. Let $R$ be of the form

$$R = \{(r, \theta) : \alpha < r < \beta, 0 \leq \theta < 2\pi\}.$$ 

Here we will work with the case where $\alpha > 0$, $\beta < \infty$ (annulus). In the textbook you can find a solved example with $\alpha = 0$ (disk), and in the homework you will solve the Dirichlet problem in the exterior of a disk, so with $\alpha > 0$ and $\beta = \infty$. In polar coordinates $(r, \theta)$, the Laplacian takes the form

$$\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}.$$ 

Start with the following warm-up problems:

**Problem 1** (Warm-up). Let $\lambda$ be a real number. We would like to find a non-trivial function $\Theta(\theta)$ satisfying the following:

$$\Theta''(\theta) + \lambda \Theta(\theta) = 0 \quad \text{for } \theta \in \mathbb{R}, \quad \Theta(\theta) = \Theta(\theta + 2\pi).$$

For what values of $\lambda$ is this possible? What are the corresponding solutions? Hint: consider separately the cases $\lambda < 0$, $\lambda = 0$, $\lambda > 0$ and find the corresponding general solutions of (1) and determine for what $\lambda$ eq. (2) can be true.

**Problem 2** (Warm-up). Find the general solution $R(r)$ to the following problem:

$$r^2 R'' + r R' = 0.$$ 

Hint: set $u = R'$ and notice that the resulting differential equation is separable.

**Problem 3** (Warm-up). Now look at the following problem:

$$r^2 R'' + r R' - n^2 R = 0,$$

where $n$ is a positive integer. This is a second order linear equation with non-constant coefficients; we have not seen a general method for solving such an equation. If you can find two linearly independent solutions $R_1$ and $R_2$, any other solution can be written as $R = AR_1 + BR_2$.

Find two linearly independent solutions in the following way: make the educated guess that the solutions will be of the form $R = r^k$, plug into the equation and find the $k$ for which the equation is satisfied.

Let $R = \{(r, \theta) : \alpha < r < \beta, 0 \leq \theta < 2\pi\}$ be an annulus as before, $\alpha > 0$. Now we would like to find $u(r, \theta)$ which is $2\pi$ periodic in $\theta$ and solves the problem

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \quad \text{on } R,$$

$$u(\alpha, \theta) = f(\theta), \quad 0 \leq \theta < 2\pi,$$

$$u(\beta, \theta) = 0, \quad 0 \leq \theta < 2\pi.$$ 

1Technically the domain becomes a punctured disk, but by assuming that your solution is continuous at the origin you are actually solving the Dirichlet problem in a disk.
We seek a solution in the form
\[ u(r, \theta) = \sum_{n=0}^{\infty} c_n u_n(r, \theta) \]
as usual. We would like to ensure that \( u_n \) satisfy (5) and (7). We will determine \( c_n \) at the end so that the sum satisfies (6) as well.

Follow the steps below to find such a solution:

1. Assume that \( u_n(r, \theta) = R_n(r) \Theta_n(\theta) \). Plug \( u_n \) into the PDE (5) and separate variables (all functions of \( r \) on the left, all functions of \( \theta \) on the right).

2. Deduce that \( \Theta_n \) satisfies an ODE of the form (1). Since we want \( u(r, \theta) = u(r, \theta + 2\pi) \), we would like the same to be true for \( u_n \). Conclude that \( \Theta_n \) also satisfies (2).

3. Use the result of Problem 1 to deduce that \( \lambda = 0 \) or \( \lambda = n^2 \), for \( n \) a positive integer.

4. Deduce that \( R_n \) must satisfy either (3) or (4). Determine the form of \( R_n \) so that \( u_n \) also satisfies (7).

5. Use the conclusions of Problems 1, 3 to deduce that
\[ u(r, \theta) = A_0 \ln \left( \frac{r}{\beta} \right) + \sum_{n=1}^{\infty} \left( \frac{r}{\beta} \right)^n - \left( \frac{r}{\beta} \right)^{-n} \right) \left( A_n \cos(n\theta) + B_n \sin(n\theta) \right) \]
\[ (8) \]

6. Now plug in \( r = \alpha \) into (8). You want (6) to be true. Recall that \( f(\theta) \) is a periodic function of period \( 2\pi \). What should the coefficients \( A_0, A_n \) and \( B_n \) be? Hint: recall that \( f \) has a Fourier series expansion.