1. Set $y = x'$. The system becomes

$$x' = y$$

$$y' = x'' = \frac{1}{m}(-cx' - kx + \beta x^3)$$

$$= \frac{1}{m}(-cy - kx + \beta x^3)$$

So: $x' = y$

$$y' = \frac{1}{m}(-cy - kx + \beta x^3)$$

2. Plug in: $\begin{cases} x' = y \\ y' = -4x - x^3 \end{cases}$

Critical pts: $y = 0$

$-4x - x^3 = 0 \Rightarrow x(x^2+4) = 0 \Rightarrow x = 0$

So only $(0,0)$. (isolated)

3. $J(x,y) = \begin{bmatrix} 0 & 1 \\ -4-3x^2 & 0 \end{bmatrix}$

At $(0,0)$: linearization $\underline{u}' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \underline{u}$

So: eigenvalues $\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i$

Origin is a center for the linearized system.

4. For the non-linear: origin can be center, spiral sink or spiral source.

5. Plug in: $\begin{cases} x' = y \\ y' = -y - 4x + x^3 \end{cases}$

a) Cp: $y = 0$

$-y - 4x + x^3 = 0 \Rightarrow x^3 - 4x = 0 \Rightarrow x = 0$

$x = \pm 2$

Cp: $(0,0)$, $(2,0)$, $(-2,0)$

b) Jacobian:

$$J(x,y) = \begin{bmatrix} 0 & 1 \\ -4+3x^2 & -1 \end{bmatrix}$$

<u>Linearization:</u>

At $(0,0)$: $\underline{u}' = \underline{J}(0,0)\underline{u} = \begin{bmatrix} 0 & 1 \\ -4 & -1 \end{bmatrix} \underline{u}$

Eigenvalues: $-\lambda(-1-\lambda) + 4 = 0$

$\Rightarrow \lambda + \lambda^2 + 4 = 0 \Rightarrow$

$\Rightarrow \lambda = \dfrac{-1 \pm i\sqrt{15}}{2}$

cplx e-values, negative im. part

$\Rightarrow$ spiral sink.

At $(2,0)$:

$$J(2,0) = \begin{bmatrix} 0 & 1 \\ 8 & -1 \end{bmatrix}$$

E-values: $-\lambda(-1-\lambda) - 8 = 0$

$\Rightarrow \lambda + \lambda^2 - 8 = 0 \Rightarrow \lambda_1 \simeq 2.37$

$\lambda_2 \simeq -3.37$

saddle

At $(-2,0)$

$$J(2,0) = \begin{bmatrix} 0 & 1 \\ 8 & -1 \end{bmatrix}$$ same as before, saddle.

c) Non-linear system : $(0,0)$ spiral sink
$(-2,0), (2,0)$ saddles

4. A: soft w/ damping
C: hard.
(B: soft w/o damping.)

5. $\begin{cases} x' = y \\ y' = -4x - x^3 \end{cases}$

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{-4x - x^3}{y}$$

$$\Rightarrow \quad y\,dy = (-4x - x^3)\,dx$$

$$\Rightarrow \int y\,dy = \int (-4x - x^3)\,dx$$

$$\Rightarrow \underbrace{\frac{1}{2}y^2 = -2x^2 - \frac{x^4}{4} + C}$$

defines the trajectories implicitly.

Can write, w/ $C > 0$

$$y = \pm \sqrt{-4x^2 - \frac{1}{2}x^4 + C}$$ traj. in figure B.

6. $$\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + \omega^2 \sin\theta = 0$$

Set $x = \theta$
$y = \theta'$

$$\begin{cases} x' = y \\ y' = -cy - \omega^2 \sin(x) \end{cases}$$

7. Plug in $c = 0$, $\omega = 1$
C.P.: $y = 0$
$\sin(x) = 0 \Rightarrow x = k\pi$, k integer.
They are far from each other (isolated).

8. $$J(x,y) = \begin{bmatrix} 0 & 1 \\ -\cos(x) & 0 \end{bmatrix}$$

If k = odd: $$J(k\pi, 0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

E-values $\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$
saddle pt.
Linearized system has saddle.

If k = even: $$J(k\pi, 0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

E-values: $\lambda = \pm i$ center.

Non-linear system: spiral sink, spiral source or center.

9: A: angle approaches $-\pi$, i.e. it approaches the position over the bolt without ever reaching it.

B: The mass revolves around the bolt

C: Mass swings periodically, away from the position over the bolt.