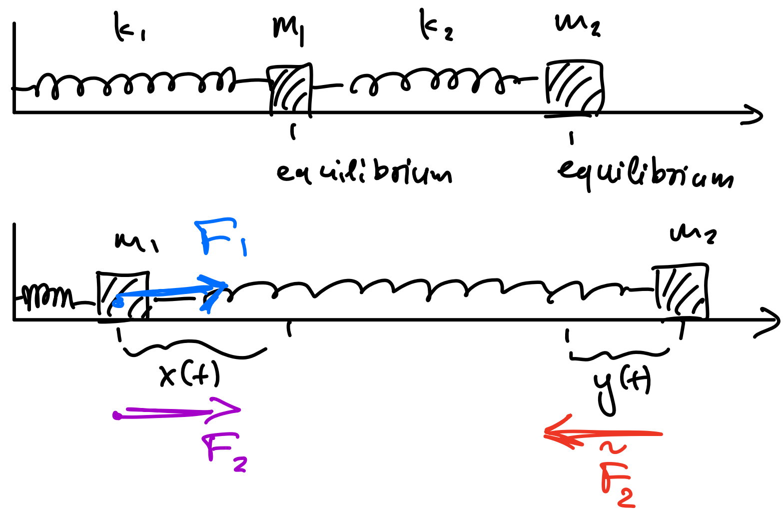
Plan:

- Write linear systems of ODE in matrix form]
- See that we can solve them
- See what solutions look like
- Do an example (if time permits)

} ?

no friction



Hooke's law

$$F_1 = -k_1 x(t)$$
$$F_2 = k_2 (y(t) - x(t))$$
$$\tilde{F}_2 = -k_2 (y(t) - x(t))$$

$$(*)\begin{cases} m_1 x''(t) = -k_1 x(t) + k_2 (y(t) - x(t)) \\ m_2 y''(t) = -k_2 (y(t) - x(t)) \end{cases}$$

Want both satisfied simultaneously: a system of order 2.

Want to turn $(*)$ into 1st order system

Rename variables.

$$u_1 = x \qquad\qquad v_1 = y$$
$$u_2 = x' \qquad\qquad v_2 = y'$$

$$u_1' = x' = u_2$$
$$u_2' = x'' = -\frac{k_1}{m_1} x + \frac{k_2}{m_1}(y-x)$$
$$= -\frac{k_1}{m_1} u_1 + \frac{k_2}{m_1}(v_1 - u_1)$$
$$v_1' = y' = v_2$$
$$v_2' = -\frac{k_2}{m_2}(y-x) = -\frac{k_2}{m_2}(v_1 - u_1)$$

System of order 1, linear.

$$* \begin{cases} u_1' = u_2 \\ u_2' = \left(-\frac{k_1}{m_1} - \frac{k_2}{m_1}\right) u_1 + \frac{k_2}{m_1} v_1 \\ v_1' = v_2 \\ v_2' = +\frac{k_2}{m_2} u_1 - \frac{k_2}{m_2} v_1 \end{cases}$$

Goal: use matrix notation for $*$

Matrix Valued Function

$$\underline{\underline{A}}(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ \vdots & \cdots & \cdots & \vdots \\ a_{m1}(t) & \cdots & \cdots & a_{mn}(t) \end{bmatrix}$$

Can differentiate entry-wise

$$\frac{d}{dt} A(t) = \left[\frac{d}{dt} a_{ij}(t)\right]$$

if c const. scalar:

$$\frac{d}{dt}\left(c\,A(t)\right) = c\,\frac{d}{dt} A(t)$$

Product rule:

$$\frac{d}{dt}\left(A(t)B(t)\right) = \left(\frac{d}{dt} A(t)\right)B(t) + A(t)\left(\frac{d}{dt} B(t)\right)$$

Ex: $A(t) = \begin{bmatrix} \sin(t) & e^t \\ t^2 & 5 \end{bmatrix}$

Linear system:

$$n \text{ eq's}\left\{\begin{array}{l} x_1'(t) = p_{11}(t)\,x_1(t) + \cdots + p_{1n}(t)\,x_n(t) + f_1(t) \\ \vdots \\ x_n'(t) = p_{n1}(t)\,x_1(t) + \cdots + p_{nn}(t)\,x_n(t) + f_n(t) \end{array}\right.$$

non-homogeneous terms

Ex:

$$\begin{cases} x_1'(t) = \sin(t)\, x_1(t) + e^t x_2(t) \\ x_2'(t) = t^2 x_1(t) + 5 x_2(t) \end{cases}$$

Non-example (non-linear)

$$\begin{cases} x_1'(t) = e^{x_1(t) + x_2(t)} \\ x_2'(t) = 2 x_2(t) \end{cases}$$

Rewrite linear system w/ matrix not.

$$\underline{\underline{x}}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad \underline{\underline{P}}(t) = \begin{bmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ & & \\ p_{n1}(t) & \cdots & p_{nn}(t) \end{bmatrix}$$

$$\underline{f}(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

So:

$$\underline{x}'(t) = \underline{\underline{P}}(t)\, \underline{\underline{x}}(t) + \underline{f}(t).$$

Check:

$$\begin{cases} x_1'(t) = x_1(t) + 2x_2(t) \\ x_2'(t) = 3x_1(t) + x_2(t) \end{cases}$$

$$\underline{x}' = \underline{\underline{A}}\, \underline{\underline{x}}, \quad A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$